SPECTRAL EXPANSIONS AND EXCURSION THEORY FOR NON-SELF-ADJOINT MARKOV SEMIGROUPS WITH APPLICATIONS IN MATHEMATICAL FINANCE

A Dissertation

Presented to the Faculty of the Graduate School

of Cornell University

in Partial Fulfillment of the Requirements for the Degree of

Doctor of Philosophy

by

Yi Xuan Zhao

December 2017

SPECTRAL EXPANSIONS AND EXCURSION THEORY FOR NON-SELF-ADJOINT MARKOV SEMIGROUPS WITH APPLICATIONS IN MATHEMATICAL FINANCE

Yi Xuan Zhao, Ph.D.

Cornell University 2017

This dissertation consists of three parts. In the first part, we establish a spectral theory in the Hilbert space $L^2(\mathbb{R}^+)$ of the C_0-semigroup P and its adjoint $\widehat{P}$ having as generator, respectively, the Caputo and the right-sided Riemann-Liouville fractional derivatives of index $1 < \alpha < 2$. These linear operators, which are non-local and non-self-adjoint, appear in many recent studies in applied mathematics and also arise as the infinitesimal generators of some substantial processes such as the reflected spectrally negative α-stable process. We establish an intertwining relationship between these semigroups and the semigroup of a Bessel type process which is self-adjoint. Relying on this commutation identity, we characterize the spectrum and the (weak) eigenfunctions and provide the spectral expansions of these semigroups on (at least) a dense subset of $L^2(\mathbb{R}^+)$. We also obtain an integral representation of their transition kernels that enables to derive regularity properties.

Inspired by this development, we further exploit, in the second part of this dissertation, the concept of intertwining between general Markov semigroups. More specifically, we start by showing that the intertwining relationship between two minimal Markov semigroups acting on Hilbert spaces implies that any recurrent extensions, in the sense of Itô, of these semigroups satisfy the same intertwining identity. Under mild additional assumptions on the inter-

twining operator, we prove that the converse also holds. This connection enables us to give an interesting probabilistic interpretation of intertwining relationships between Markov semigroups via excursion theory: two such recurrent extensions that intertwine share, under an appropriate normalization, the same local time at the boundary point. Moreover, in the case when one of the (non-self-adjoint) semigroup intertwines with the one of a quasi-diffusion, we obtain an extension of Krein's theory of strings by showing that its densely defined spectral measure is absolutely continuous with respect to the measure appearing in the Stieltjes representation of the Laplace exponent of the inverse local time. Finally, we illustrate our results with the class of positive self-similar Markov semigroups and also the reflected generalized Laguerre semigroups. For the latter, we obtain their spectral decomposition and provide, under some conditions, a perturbed spectral gap estimate for its convergence to equilibrium.

The third part of this dissertation is devoted to the applications of some of these theoretical results to some substantial problems arising in financial mathematics. Keeping in mind the fundamental theorem of asset pricing, we suggest several transformations on a tractable and flexible Markov process (or equivalently, its respective semigroup) in order that the discounted transformed process becomes a (local) martingale while still keeping its tractability. In particular, we suggest using an intertwining approach and/or Bochner's subordination (random time-change via a subordinator) to achieve this goal. Moreover, in order to illustrate our approach, we discuss in details several examples that include the class of Lévy, self-similar and generalized CIR processes that reveal the usefulness of our result. Furthermore, we provide for the non-self-adjoint pricing semigroups associated to the latter family of processes a spectral expansions on which we carry out some numerical analysis.

BIOGRAPHICAL SKETCH

Yi Xuan Zhao was born and raised in Dalian, a modern coastal city in northeast China which serves as an important naval base and logistics centre. He attended University of Waterloo studying mathematical finance and operations research, after which he continued to pursue a doctoral degree in the ORIE department at Cornell University.

During his five years spent at Cornell, Yi Xuan worked with Professor Pierre Patie on properties of some non-self-adjoint Markov semigroups, mainly focusing on their spectral properties, excursion theory and their applications in mathematical finance. Besides his academic work, he enjoys traveling around the world as much as spending time with his friends in cinema, theatres and restaurants.

After his graduation, Yi Xuan will join Credit Suisse in New York City as a quantitative strategist.

This document is dedicated to all Cornell graduate students.

ACKNOWLEDGEMENTS

My deepest gratitude goes to my advisor, Pierre Patie, for his guidance and support over the five years in Cornell. He motivates me not only in academic studies, but also in all aspects in life. His immense knowledge and great patience have deeply impressed me, and his rigorous attitude towards work has influenced me so much. It would not be possible for me to achieve my academic performance without his help and encouragement.

I would also like to thank my committee members, Robert Jarrow and Laurent Saloff-Coste, for offering their help in my research. Thank all the faculty and staff at the Department of ORIE, especially Krishnamurthy Iyer, Andreea Minca, Sidney Resnick, Gennady Samorodnitsky and Bruce Turnbull, with whom I have had the great pleasure to work as a teaching assistant. Special thanks to Peter Frazier, who offered me great help in my memorable experience working as the instructor of the Monte Carlo simulation course.

Next, I thank all my Ph.D. fellow and friends at Cornell. Thank you my office mates Jiayang Gao, Nanjing Jian, Jiayi Guo and Eric Ni, with whom I share so many stories in life and so much cherishable memories and pleasure. Also thank Weici Hu, Bangrui Chen, Tiandong Wang, Sijia Ma, Michael Choi and Amy Zhang, for all your company and help.

Last but not the least, I am grateful to my parents Hongxing Zhao and Mei Han, for their uoconditional love and support.

TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION

The spectral theory of linear operators has been an essential subject in functional analysis. First introduced by David Hilbert in his original formulation of Hilbert space theory, it has revealed its importance in various fields of mathematics, including differential equations, probability theory, harmonic and complex analysis, etc. Moreover, it is also widely used in a large number of different application areas, such as superconductivity, fluid mechanics, quantum mechanics, kinetic theory, and more recently, financial mathematics.

In functional analysis, the motivation for studying spectral theory is to understand the structure of a linear operator. Such a structure often includes, but is not restricted to, classifying the operator by means of equivalent transformations, reconstructing the operator in simple forms such as a direct sum or direct integral, identifying invariant subspaces, characterizing a basis for its domain or range, etc. For finite dimensional matrices, these problems can be solved by means of eigenvalues and their corresponding eigenvectors. While in the case of a infinite-dimensional functional operator, the idea of eigenvalues generalizes into the so-called spectrum, whose formal definition will be given later in the context. Moreover, spectral theory enables us to classify a bounded normal operator by means of unitary equivalence, or represent it as an integral with respect to projection-valued measures over its spectrum, i.e. the resolution of identity. Such results are analogous to the eigendecomposition for finite dimensional matrices.

However, the classical spectral theory is mainly focused on normal operators in Hilbert spaces. For the class of non-self-adjoint (NSA) operators, on the contrary, there has been only limited results, due to their non-local and non-normal nature, which brings significant difficulty in studying their spectral properties. In a recent work by Patie and Savov [91], the authors suggest an original approach to tackle this problem for a certain class of non-local and non-self-adjoint operators, that they name the generalized Laguerre semigroups. Their main idea stems on an intertwining relationship that they establish between each element of this class and the classical self-adjoint Laguerre semigroup $(Q_t)_{t\geq 0}$, whose spectral properties have been well studied. In other words, they managed to show that for each semigroup $(P_t)_{t\geq 0}$ in the class of generalized Laguerre semigroups there exists an intertwining kernel Λ, such that for any $t \geq 0$,

$$P_t\Lambda = \Lambda Q_t \tag{1.1}$$

where the identity holds on a weighted Hilbert space. Using this intertwining relationship, they successfully characterized a sequence of eigenfunctions and co-eigenfunctions (that is, the eigenfunctions for the Hilbert space adjoint semigroup), and provided a eigenvalues expansion for such type of non-self-adjoint generalized Laguerre semigroups under various conditions.

This intertwining idea in [91] naturally inspires the first issue of this dissertation. Besides the generalized Laguerre semigroups, does there exist such type of intertwining relationship between other NSA semigroups and self-adjoint ones? Can we apply this method to study the spectral properties of some other NSA semigroups that are of significant importance in modeling and applications? Moreover, note that in [91], the intertwined semigroups P and Q both have 0 as an entrance-no-exit boundary. Therefore, it is natural to ask whether there exist intertwined pairs of semigroups with a common killing boundary (i.e. with a Dirichlet boundary condition), or reflecting boundary (Neumman boundary condition), or a combination of both (Robin boundary condition)? If so, it is more interesting to ask that in a more general sense, given two intertwined semigroups, both killed at a common boundary, does it imply the same type of intertwining relationship between their extensions? What about the converse?

For these issues, noticing that the construction of Markovian extensions of minimal (killed) semigroup is based on excursions theory, we aim at establishing a connection of this latter theory and intertwining relationships. To this end, let us briefly recall that, as initiated by Itô [59], the purpose of excursion theory is to describe the evolution of a strong Markov process X in terms of its behavior between visits to a specific regular point b in its state space. The excursions from b are then pieces of paths, starting and ending at b, each of random lengths which are almost surely finite. Moreover, the path of the original process can be reconstructed from the excursions and the time spent at b, the latter called the "local time" at b. Its inverse process, which is known to be a Lévy subordinator, that is a non-decreasing real-valued continuous time process with stationary and independent increments, provides a convenient way to label the set of all excursions from b. Therefore, by considering two Markov processes whose respective semigroups intertwine and share a common regular point b, our next issue of this dissertation is whether this intertwining relation can be explained by excursion theory. In particular, does there exist a relationship between their local times, excursion lengths, excursion laws, etc?

In fact, there is already, for one-dimensional quasi-diffusions, a known and beautiful connection between the spectral and excursion theory which is the celebrated Krein's spectral theory of strings, see e.g. [63, 65]. In short, Krein provides a bijection between the inverse local times (at a regular point b) of quasi-diffusions and the spectral measures of their corresponding semigroups killed at b. We will exlain the Krein's result in detail later in this dissertation, but nevertheless, it motivates us to combine our discovery in the spectral expansions and inverse local times, again for non-diffusion processes, to seek a potential extension for the classical Krein's theory.

Last but not the least, we are interested in applying our results, especially the idea of intertwining relationship and the spectral expansions for non-self-adjoint semigroups, to some substantial issues arising in financial mathematics. We shall recall from the fundamental theorem of asset pricing that a market admits no free lunch if and only if there exists an equivalent martingale measure under which all discounted risky asset prices are (local) martingales. We relate this requirement with the notion of extended generators as presented in [36] via Dynkin's theorem. In particular, a Markov process X is a martingale if and only if $\widetilde{A}p_1 = 0$, where $\widetilde{A}$ is the extended generator of X and $p_1(x) = x$ is the identity function. However, although some classes of tractable stochastic processes have very nice features that we would like to incorporate into financial modeling, these processes often do not, by themselves, form (local) martingales under the current measure. Therefore, the problem of transforming the original process (or its corresponding semigroup) in order to make it a martingale now becomes essential. We call this procedure *risk-neutral pricing techniques*. In particular, for some classes of non-self-adjoint semigroups whose spectral expansions can be represented in a simple form, we can easily compute its value numerically and provide an approximation with high accuracy.

This dissertation, solving all the above issues, consists of four chapters. Besides this current chapter of introduction and preliminaries, each of the remaining three chapters are based on published or submitted papers and unpublished manuscripts. Chapter 2 is based on "Spectral decomposition of fractional operators and a reflected stable semigroup", published in *Journal of Differential Equations*. It focuses on studying the spectral properties of some fractional operators, namely the Riemann-Liouville derivative and the Caputo fractional derivative. These operators play an important role in many areas, such as population dynamics, chemical technology, biotechnology and control of dynamical systems. Since these fractional operators indeed serve as the infinitesimal generators for stable related semigroups, we focus on studying these semigroups instead of the operators directly. Our approach relies on intertwining relations that we establish between these semigroups and the semigroup of a Bessel type process whose generator is a self-adjoint second order differential operator. In particular, from this commutation relation, we characterize the positive real axis as the continuous point spectrum of P and provide a power series representation of the corresponding eigenfunctions. We also identify the positive real axis as the residual spectrum of the adjoint operator $\widehat{P}$ and elucidates its role in the spectral decomposition of these operators. By resorting to the concept of continuous frames, we proceed by investigating the domain of the spectral operators and derive two representations for the heat kernels of these semigroups. As a by-product, we also obtain regularity properties for these latter and also for the solution of the associated Cauchy problem.

Chapter 3 is based on the paper "Intertwining, excursion theory and Krein theory of strings for non-self-adjoint Markov semigroups". In this chapter, we study the intertwining relationship between general non-self-adjoint and non-local Markov semigroups, and interpret this relationship via excursion theory. In particular, assuming that there exists a common regular point b for two strong Markov processes, we show that under certain conditions, the intertwining relation between the minimal semigroups (killed at the first hitting time of b) implies the same relation between their extensions, and the converse also holds under additional assumptions on the intertwining kernel. We also show that under this intertwining relation, the Laplace exponents of the inverse local time at b for these two processes coincide. Relying on this observation, we offer an extension of Krein's theory of strings to non-diffusions by relating the (density of the) Lévy measure of the inverse local time to the Laplace transform of a weak version of the spectral measure, which is defined only on a dense subset, of the minimal semigroup. Furthermore, we illustrate these ideas by studying the class of (non-self-adjoint) self-similar and reflected (at 0) generalized Laguerre semigroups for which we show that they satisfy the extended Krein property by characterizing their spectral expansions and the Laplace exponent of their inverse local times.

Chapter 4 is based on the manuscript "Risk-neutral pricing techniques and examples", which is a joint work with Robert Jarrow, Pierre Patie and Anna Srapionyan. Starting with a general Markov process whose discounted value is not necessarily a martingale, we suggest several methods to transform the process or its corresponding semigroup such that, after the transformation, the identity function is r-invariant, while still remaining its tractability. The first method is via an intertwining relation, from which we deduce two special cases, one using an invariant function and the other using Doob's h-transform. The second method is to perform a Bochner's subordination to the process, i.e. a random time change according to a Lévy subordinator. We further provide a few examples, illustrating some classes of Markov processes where our techniques find useful. We mention that these methods can be used to solve various problems in mathematical finance, e.g. modifying the Merton's structural model, interpreting a firm's stock price as a function of macro-economic factors, modeling an asset's liquidity by means of random time-change, etc.

At the end of this introduction, we shall introduce some notations and preliminary knowledge that will be used throughout the dissertation, as presented in the following section.

1.1 Notations and preliminaries

Let $(E, \mathcal{E})$ be a Lusin state space, with $\mathrm{B}_b(E)$ (resp. $\mathrm{B}_b^+(E)$) denote the space of bounded real-valued (resp. bounded real-valued and non-negative) measurable functions on E, $\mathsf{C}_b(E)$ denote the space of bounded continuous functions on E. We also write $\mathrm{L}^2(E)$ for the Hilbert space of square integrable Lebesgue measurable functions on E endowed with the inner product $\langle f, g\rangle = \int_E f(x)g(x)dx$ and the associated norm $\|\cdot\|$. For any weight function m defined on E, i.e. a non-negative Lebesgue measurable function, we denote by $\mathrm{L}^2(m)$ the weighted Hilbert space endowed with the inner product $\langle f, g\rangle_m = \int_E f(x)g(x)m(x)dx$ and its corresponding norm $\|\cdot\|_m$. We also denote

$$(f, g)_m = \int_E f(x)g(x)m(dx) \tag{1.2}$$

Whenever this integral exists.

In particular, when $E = \mathbb{R}_+ = (0, \infty)$ is the positive half-line, we let $\mathrm{C}_0(\mathbb{R}_+)$ denote the space of continuous real-valued functions on $\mathbb{R}_+$ tending to 0 at infinity, which becomes a Banach space when endowed with the uniform topology $\|\cdot\|_\infty$. Additionally, we denote $\mathrm{C}_0^2(\mathbb{R}_+)$ to be the space of twice continuously differentiable functions on $\mathbb{R}_+$, which vanishes at both 0 and infinity, and $\mathrm{C}^\infty(\mathbb{R}_+)$ the space of functions with continuous derivatives on $\mathbb{R}_+$ of all orders.

For any $-\infty \leq \underline{a} < \overline{a} \leq \infty$, we denote the strip $\mathbb{C}_{(\underline{a},\overline{a})} = \{z \in \mathbb{C};\ \underline{a} < \Re(z) < \overline{a}\}$, and write simply $\mathbb{C}_+ = \mathbb{C}_{[0,\infty)}$. We write $\mathbb{C}_{(-\infty,0)^c} = \{z \in \mathbb{C};\ \arg(z) \neq \pi\}$ for the complex plane cut along the negative real axis.

For Banach spaces $\mathrm{H}_1, \mathrm{H}_2$, we define

$$\mathbf{B}(\mathrm{H}_1, \mathrm{H}_2) = \{L : \mathrm{H}_1 \to \mathrm{H}_2 \text{ linear and continuous mapping}\}.$$

In the case of one Banach space H, the unital Banach algebra $\mathbf{B}(\mathrm{H}, \mathrm{H})$ is simply denoted by $\mathbf{B}(\mathrm{H})$. Moreover, a semigroup $P = (P_t)_{t\geq 0}$ where $P_t \in \mathbf{B}(\mathrm{H})$ is called a positive C_0-semigroup on H if $P_{t+s} = P_t \circ P_s$, $P_t f \geq 0$ for $f \geq 0$, and for any functions $f \in \mathrm{H}$, $\|P_t f - f\|_{\mathrm{H}} \to 0$ as $t \to 0$. In the case when $\mathrm{H} = \mathrm{C}_0(\mathbb{R}_+)$ endowed with the uniform topology, we say P is a Feller semigroup on $\mathbb{R}_+$. Furthermore, for an operator $T \in \mathbf{B}(\mathrm{H}_1, \mathrm{H}_2)$, we use the notation $\mathrm{Ran}(T)$ (resp. $\mathrm{Ker}(T)$) for the range (resp. the kernel) of T and $\overline{\mathrm{Ran}}(T)$ (resp. $\overline{\mathrm{Ker}}(T)$) for its closure. For any set of functions $\mathrm{E} \subseteq \mathrm{H}$, we use $\mathrm{Span}(\mathrm{E})$ to denote the set of all linear combinations of functions in E, and $\overline{\mathrm{Span}}(\mathrm{E})$ for its closure.

For two functions $f, g : \mathbb{R}_+ \to \mathbb{R}$, we write $f \stackrel{a}{=} \mathrm{O}(g)$ (resp. $f \stackrel{a}{=} \mathrm{o}(g)$) if $\limsup_{x\to a} \frac{f(x)}{g(x)} < \infty$ (resp. $\lim_{x\to a} \frac{f(x)}{g(x)} = 0$), and $f \asymp g$ (resp. $f \stackrel{a}{\sim} g$) if $\exists\, c > 0$ such that $c \leq \frac{f(x)}{g(x)} \leq c^{-1}$ for all $x \in \mathbb{R}_+$ (resp. if $\lim_{x\to a} \frac{f(x)}{g(x)} = 1$ for some $a \in \mathbb{R} \cup \{\pm\infty\}$). Furthermore, for any $q \in \mathbb{R}_+$, we write $\mathrm{d}_q f(x) = f(qx)$ the dilation operator. Next, let P be a bounded linear operator acting on a Hilbert space H over $\mathbb{C}$, and let I denote the identity operator on H, then we use $\sigma(\mathrm{P})$ to denote the spectrum of P, which is defined by

$$\sigma(\mathrm{P}) = \{\lambda \in \mathbb{C};\ \mathrm{P} - \lambda I \text{ does not have a bounded inverse in } H\},$$

with the following three distinctions:

- $\lambda \in \sigma_p(\mathrm{P})$, the point spectrum, if $\mathrm{Ker}(\mathrm{P} - \lambda I) \neq \{0\}$. In this case, we say a function f_λ is an eigenfunction for P, associated to the eigenvalue λ, if $f_\lambda \in \mathrm{Ker}(\mathrm{P} - \lambda I)$.
- $\lambda \in \sigma_c(\mathrm{P})$, the continuous spectrum, if $\mathrm{Ker}(\mathrm{P} - \lambda I) = \{0\}$ and $\overline{\mathrm{Ran}}(\mathrm{P} - \lambda I) = H$ but $\mathrm{Ran}(\mathrm{P} - \lambda I) \subsetneq H$.
- $\lambda \in \sigma_r(\mathrm{P})$, the residual spectrum, if $\mathrm{Ker}(\mathrm{P} - \lambda I) = \{0\}$ and $\overline{\mathrm{Ran}}(\mathrm{P} - \lambda I) \subsetneq H$.

CHAPTER 2

SPECTRAL DECOMPOSITION OF FRACTIONAL OPERATORS AND A REFLECTED STABLE SEMIGROUP

2.1 Introduction

Fractional calculus, in which derivatives and integrals of fractional order are defined and studied, is nearly as old as the classical calculus of integer orders. Ever since the first inquisition by L'Hopital and Leibniz in 1695, there has been an enormous amount of study on this topic for more than three centuries, with many mathematicians having suggested their own definitions that fit the concept of a non-integer order derivative. Among the most famous of these definitions are the Riemann-Liouville fractional derivative and the Caputo derivative, the latter being a reformulation of the former in order to use integer order initial conditions to solve fractional order differential equations. In this context, it is natural to consider the following Cauchy problem, for a smooth function f on $x > 0$,

$$\begin{cases} \frac{d}{dt}u(t,x) = \mathbf{D}_\alpha u(t,x) \\ u(0,x) = f(x), \end{cases} \tag{2.1}$$

where, for any $1 < \alpha < 2$, the linear operator $\mathbf{D}_\alpha$ is either the Caputo α-fractional derivative

$$\mathbf{D}_\alpha f(x) = {}^{C}D_{+}^{\alpha} f(x) = \int_0^x \frac{f^{([\alpha]+1)}(y)}{(x-y)^{\alpha-[\alpha]}} \frac{dy}{\Gamma([\alpha]+1-\alpha)}, \tag{2.2}$$

with, for any $k = 1, 2, \ldots$, $f^{(k)}(x) = \frac{d^k}{dx^k} f(x)$ stands for the k-th derivative of f, or, the right-sided Riemann-Liouville (RL) derivative

$$\mathbf{D}_\alpha f(x) = D_{-}^{\alpha} f(x) = \left(\frac{d}{dx}\right)^{[\alpha]+1} \int_x^\infty \frac{f(y)(y-x)^{[\alpha]-\alpha}}{\Gamma([\alpha]+1-\alpha)} dy, \tag{2.3}$$

with $[\alpha]$ representing the integral part of α. We point out that when $\alpha = 2$, in both cases, $\mathbf{D}_2 f(x) = \frac{1}{2} f^{(2)}(x)$ is a second order differential operator.

In this paper, we aim at providing the spectral representation in $L^2(\mathbb{R}_+)$ Hilbert space and regularities properties of the solution to the Cauchy problem (2.1).

The motivation underlying this study are several folds. On the one hand, the last three decades have witnessed the most intriguing leaps in engineering and scientific applications of such fractional operators, including but not limited to population dynamics, chemical technology, biotechnology and control of dynamical systems, and, we refer to the monographs of Kilbas et al. [61], Meerschaert and Sikorskii [74] and Sankaranarayanan [105] for excellent and recent accounts on fractional operators. On the other hand, some recent interesting studies have revealed that the linear operator ${}^C D_+^\alpha$ is the infinitesimal generator of $P = (P_t)_{t \geq 0}$ the Feller semigroup corresponding to the so-called spectrally negative reflected α-stable process, see e.g. [6, 12, 92]. We will provide the formal definition of this process and semigroup in Section 2.2, and, we simply point out that the reflected Brownian motion is obtained in the limiting case $\alpha = 2$. The reflected α-stable processes have been studied intensively in the stochastic processes literature. In particular, we mention that, in a recent paper, Baeumer et. al. [6] showed the interesting fact that the transition kernel of P allows to map the set of solutions of a Cauchy problem to its fractional (in time) analogue. Motivated by these findings, they provide a numerical method to approximate this transition kernel. In this perspective, in Theorem 2.6.2 below, we provide two analytical and simple expressions for this transition kernel.

Although the Cauchy problem for the fractional operators associated to reflected stable processes plays a central role in many fields of sciences, to the best of our knowledge, their spectral representation remain unclear. This seems to be attributed to the fact that there is not a unified theory for dealing with the spectral decomposition of non-local and non-self-adjoint operators, two properties satisfied, as we shall see in Proposition 2.2.1, by the fractional operators considered therein. For a nice account on classical and recent developments on this important topic, we refer to the two volume treatise of Dunford and Schwartz [45, 46] and the monograph of Davies [41], and the survey paper by Sjöstrand [110].

The purpose of this paper is to provide detailed information regarding the solution of the Cauchy problem (2.1) along with its elementary solution which corresponds to the transition probabilities of the Feller semigroups P and its dual $\widehat{P}$. More specifically, we provide a spectral representation of this solution in an integral form involving the absolutely continuous part of the spectral measure, the generalized Mittag-Leffler functions as eigenfunctions and a weak Fourier kernel, a terminology which is defined in [87] and recalled in Section 2.5. This kernel admits on a dense subset an integral representation which is given in terms of a function, having a simple expression, that we name a residual function for the dual semigroup (or co-residual function for P), as it is associated to elements in its residual spectrum. We refer to Section 2.5 for more precise definitions. As by-product of this spectral representation, we manage to derive regularity properties for the solution of (2.1) and also for the transition kernel. We already mention that we observe a cut-off phenomenon in the nature of the spectrum for the class of operators indexed by the parameter $\alpha \in (1, 2]$. Indeed, while the class of Bessel operators which include the limit case $\mathbf{D}_2$, i.e. $\alpha = 2$, has the positive axis $(0, \infty)$ as continuous spectrum, we shall show that this axis corresponds, when $\alpha \in (1,2)$, to the continuous point spectrum of the Caputo operator and the residual spectrum of the right-sided RL fractional operator.

Our approach relies on an in-depth analysis of an intertwining relation that we establish between the Caputo fractional operator and a second order differential operator of Bessel type, which the latter turns out to be the generator of a self-adjoint semigroup in $L^2(\mathbb{R}_+)$. This is combined with the theory of continuous frames that have been introduced recently in the mathematical physics literature, see [2]. This work complements nicely the recent works of Patie and Savov in [88] and [91] where such ideas are elaborated between linear operators having a common discrete point spectrum. We also mention that recently Kuznetsov and Kwasnicki [66] provide a representation of the transition kernel of α-stable processes killed upon entering the negative real line, by inverting their resolvent density that they manage to compute explicitly. In this vein but in a more general context, Patie and Savov in the work in progress [87] explore further the idea developed in our paper to establish the spectral theory of the class of positive self-similar semigroups.

The rest of this chapter is organized as follows. In Section 2.2, we introduce the reflected one-sided α-stable processes and establish substantial analytical properties of the corresponding semigroups. In Section 2.3, we shall derive the intertwining relation between the spectrally negative reflected stable semigroup and the Bessel-type semigroup. From this link, we extract a set of eigenfunctions that are described in Section 2.4 which also includes some of their interesting properties such as the continuous upper frame property, completeness and large asymptotic behavior. In Section 2.5 we investigate the so-called co-residual functions. Finally, in Section 2.6 we gather all previous results to provide the spectral decomposition of the two semigroups P and $\widehat{P}$ including two representations for their transition kernels. The regularity properties are also stated and proved in that Section.

2.2 Fractional operators and the reflected stable semigroup

Let $Z = (Z_t)_{t\geq 0}$ be a spectrally negative α-stable Lévy process with $\alpha \in (1,2)$, defined on a filtered probability space $(\mathbf{\Omega}, \mathbf{F}, (\mathbf{F}_t)_{t\geq 0}, \mathbb{P} = (\mathbb{P}_x)_{x\in\mathbb{R}})$. It means that Z is a process with stationary and independent increments, having no positive jumps, and its law is characterized, for $t > 0$, by

$$\log \mathbb{E}[e^{zZ_t}] = z^\alpha t, \quad z \in \mathbb{C}_+. \tag{2.4}$$

Here and below z^α is the main branch of the complex analytic function in the complex half-plane $\Re(z) \geq 0$, so that $1^\alpha = 1$. Let $X = (X_t)_{t\geq 0}$ be the process Z reflected at its infimum, that is, for any $t \geq 0$,

$$X_t = \begin{cases} Z_t & \text{if} \quad t < T^Z_{(-\infty,0]}, \\ Z_t - \inf_{s\leq t} Z_s & \text{if} \quad t \geq T^Z_{(-\infty,0]}, \end{cases}$$

with $T^Z_{(-\infty,0]} = \inf\{t > 0;\ Z_t \leq 0\}$, and we write, for any $f \in \mathrm{B}_b(\mathbb{R}_+), t, x \geq 0$,

$$P_t f(x) \quad = \quad \mathbb{E}_x\left[f(X_t)\right], \tag{2.5}$$

where $\mathbb{E}_x$ stands for the expectation operator associated to $\mathbb{P}_x(Z_0 = x) = 1$. Next, let $\widehat{Z} = -Z$ be the dual process of Z (with respect to the Lebesgue measure), which is a spectrally positive α-stable process, and, let $\widehat{X} = (\widehat{X}_t)_{t\geq 0}$ be the process defined from $\widehat{Z}$ by a random time-change as follows, for any $t \geq 0$,

$$\widehat{X}_t = \widehat{Z}_{\hat{\tau}_t}, \tag{2.6}$$

where $\hat{\tau}_t = \inf\{u > 0;\ \widehat{A}_u > t\}$ and $\widehat{A}_t = \int_0^t \mathbb{I}_{\{\widehat{Z}_s>0\}} ds$. We also write for any $f \in \mathrm{B}_b(\mathbb{R}_+), t, x \geq 0$,

$$\widehat{P}_t f(x) = \widehat{\mathbb{E}}_x[f(\widehat{X}_t)],$$

where $\widehat{\mathbb{E}}_x$ stands for the expectation operator associated to $\widehat{\mathbb{P}}_x(\widehat{Z}_0 = x) = 1$. We are now ready to state our first result.

Proposition 2.2.1. 1. *P is a positive contractive C_0-semigroup on $\mathrm{C}_0(\mathbb{R}_+)$, i.e. a Feller semigroup, whose infinitesimal generator is $({}^C D_+^{\alpha}, \mathcal{D}_\alpha)$ where*

$$\mathcal{D}_\alpha = \left\{ f \in \mathrm{C}_0(\mathbb{R}_+);\ f(x) = \int_0^\infty \left(e^{-y}\mathcal{J}_\alpha(x) - \mathcal{J}_\alpha^{'}(x-y)\mathbb{I}_{\{y<x\}} \right) g(y)dy, g \in \mathrm{C}_0(\mathbb{R}_+) \right\},$$

with

$$\mathcal{J}_\alpha(z) \quad = \quad \frac{1}{\Gamma(1+\frac{1}{\alpha})} \sum_{n=0}^{\infty} \frac{(e^{i\pi} z^\alpha)^n}{\Gamma(\alpha n + 1)}, \quad z \in \mathbb{C}, \tag{2.7}$$

which is easily seen to define a function holomorphic on $\mathbb{C}_{(-\infty,0)^c}$.

2. *P admits a unique extension as a contractive C_0-semigroup on $\mathrm{L}^2(\mathbb{R}_+)$, which is also denoted by $P = (P_t)_{t\geq 0}$ when there is no confusion (otherwise we may denote P^F for the Feller semigroup). The domain of its infinitesimal generator L^X is given by*

$$\mathcal{D}_\alpha(\mathrm{L}^2(\mathbb{R}_+)) = \left\{ f \in \mathrm{L}^2(\mathbb{R}_+);\ \int_{-\infty}^{\infty} \left| \mathcal{F}_f^+(\xi) \right|^2 |\xi|^{2\alpha} d\xi < \infty \right\} \tag{2.8}$$

where $\mathcal{F}_f^+(\xi) = \int_0^\infty e^{i\xi x} f(x)dx$ is the one-sided Fourier transform of f taken in the L^2 sense.

3. *$\widehat{X}$ is the (weak) dual of X with respect to the Lebesgue measure. Moreover, $\widehat{P}$ is a Feller semigroup which admits a unique extension as a contractive C_0-semigroup on $\mathrm{L}^2(\mathbb{R}_+)$, also denoted by $\widehat{P}$, which has $(D_-^\alpha, \mathcal{D}_\alpha(\mathrm{L}^2(\mathbb{R}_+)))$ as infinitesimal generator. Clearly as $P \neq \widehat{P}$, we get that P is non-self-adjoint in $\mathrm{L}^2(\mathbb{R}_+)$.*

Remark 2.2.1. *We point out that when $\alpha = 2$, P is the 1-dimensional Bessel semigroup, see [23, Appendix 1], which also belongs to the class of the so-called α-Bessel* *semigroups, which are reviewed in more details in Appendix 2.7. In this case,* $\widehat{P} = P$ *and* P *is self-adjoint in* $\mathrm{L}^2(\mathbb{R}_+)$.

Remark 2.2.2. *Note that the function* $\mathcal{J}_\alpha(e^{i\pi}z^{\frac{1}{\alpha}})$ *is the (generalized) Mittag-Leffler function of parameters* $(\alpha, 1)$*, see e.g. [61] for a detailed account on this function.*

In order to prove this Proposition, we first state and prove the following lemma, which generalizes [13, Lemma 2] and may be of independent interests.

Lemma 2.2.1. *Let* $Y_t = Z_{\tau_t}, t \geq 0$*, where* $\tau_t = \inf\{u > 0;\ A_u > t\}$ *and* $A_t = \int_0^t \mathbb{I}_{\{Z_s > 0\}} ds$*. Then* $(Y_t)_{t\geq 0}$ *is a* $(\mathbf{F}_{\tau_t})_{t\geq 0}$ *strong Markov process and for any* $f \in \mathrm{B}_b(\mathbb{R}_+), t, x \geq 0$*, we have*

$$P_t f(x) = \mathbb{E}_x[f(Y_t)]. \tag{2.9}$$

Moreover, $(Y_t)_{t\geq 0}$ *and* $(\widehat{X}_t)_{t\geq 0}$ *are dual processes with respect to the Lebesgue measure.*

Proof. For any $f \in \mathrm{B}_b(\mathbb{R}_+), q > 0$, let

$$U_q f(x) = \int_0^\infty e^{-qt} P_t f(x) dt, \quad U_q^0 f(x) = \int_0^\infty e^{-qt} \mathbb{E}_x \left[f(X_t) \mathbb{I}_{\{t < T_0^X\}} \right] dt$$

be the resolvents of X and $X^0 = (X_t^0)_{t\geq 0}$, the process X killed at time $T_0^X = \inf\{t > 0;\ X_t = 0\}$, respectively. It is easy to observe from the construction of X that $T_0^X = T_{(-\infty,0]}^Z$. Moreover, by [101, Example 3], X can also be defined as the unique self-similar recurrent extension of X^0 and we get, from an application of the strong Markov property, that for all $x \geq 0$,

$$U_q f(x) = U_q^0 f(x) + \mathbb{E}_x \left[e^{-qT_0^X} \right] U_q f(0). \tag{2.10}$$

Next, since Z has paths of unbounded variation, by [67, Theorem 6.5], we have $\mathbb{P}_x(T_{[0,\infty)}^Z = 0) = 1$ for $x \geq 0$ and $\mathbb{P}_x(T_{[0,\infty)}^Z > 0) = 1$ for any $x < 0$, where $T_{[0,\infty)}^Z = \inf\{t > 0; Z_t \geq 0\}$. Thus, the fine support of the additive functional $(A_t)_{t\geq 0}$, defined as the set $\{x \in \mathbb{R};\ \mathbb{P}_x(\tau_0 = 0) = 1\}$, is plainly $[0, \infty)$. Moreover, as the Lévy process Z is a Feller process and therefore a Hunt process (see e.g. [35, Section 3.1]), we have from [56] that $(Y_t)_{t\geq 0}$ is a $(\mathbf{F}_{\tau_t})_{t\geq 0}$ strong Markov process, whose resolvent is defined, for $f \in \mathrm{B}_b(\mathbb{R}_+)$, by

$$V_q f(x) = \int_0^\infty e^{-qt}\mathbb{E}_x[f(Y_t)]dt.$$

Furthermore, it is easy to observe that $A_t = t$ for any $t \leq T^Z_{(-\infty,0]}$ and thus $\tau_t = t$ for any $t < T^Z_{(-\infty,0]}$. On the other hand, since Z is a spectrally negative Lévy process with no Gaussian component, Z does not creep below, see e.g. [67, Exercise 7.4], and therefore $T^Z_0 = \inf\{t > 0;\ Z_t = 0\} > T^Z_{(-\infty,0]}$ *a.s.*, where *a.s.* throughout this proof, means $\mathbb{P}_x$-almost surely for all $x > 0$. Moreover, observe that a.s.

$$A_{T^Z_0} = \int_0^{T^Z_{(-\infty,0]}} \mathbb{I}_{\{Z_s>0\}}ds + \int_{T^Z_{(-\infty,0]}}^{T^Z_0} \mathbb{I}_{\{Z_s>0\}}ds = A_{T^Z_{(-\infty,0]}} = T^Z_{(-\infty,0]}.$$

Next, recalling that $T^Z_{(-\infty,0]} = T^X_0$, we deduce from the previous identity that, with the obvious notation, a.s.

$$T^Y_0 = A_{T^Z_0} = T^Z_{(-\infty,0]} = T^X_0. \tag{2.11}$$

Since it is clear that $Y_t = Z_{\tau_t} = Z_t = X_t$ for $t < T^X_0$, we have for any $f \in \mathrm{B}_b(\mathbb{R}_+)$ and $q > 0$,

$$V^0_q f(x) = \int_0^\infty e^{-qt}\mathbb{E}_x\left[f(Y_t)\mathbb{I}_{\{t<T^Y_0\}}\right]dt = \int_0^\infty e^{-qt}\mathbb{E}_x\left[f(X_t)\mathbb{I}_{\{t<T^X_0\}}\right]dt = U^0_q f(x).$$

Hence, the strong Markov property of $(Y_t)_{t\geq 0}$ together with (2.11) yield that, for every $x \geq 0$,

$$V_q f(x) = V^0_q f(x) + \mathbb{E}_x\left[e^{-qT^Y_0}\right]V_q f(0) = U^0_q f(x) + \mathbb{E}_x\left[e^{-qT^X_0}\right]V_q f(0).$$

Next, according to [13, Lemma 2] and after an obvious dual argument, $(Y_t)_{t\geq 0}$ and $(X_t)_{t\geq 0}$ have the same law under $\mathbb{P}_0$ and therefore $V_q f(0) = U_q f(0)$. Hence

$$U_q f(x) = U^0_q f(x) + \mathbb{E}_x\left[e^{-qT^X_0}\right]U_q f(0) = U^0_q f(x) + \mathbb{E}_x\left[e^{-qT^X_0}\right]V_q f(0) = V_q f(x),$$

which proves the identity (2.9). Next, by [116, Proposition 4.4], we observe that $(A_t)_{t\geq 0}$ and $(\widehat{A_t})_{t\geq 0}$ are dual additive functionals, both of which are finite for each t and continuous. Hence by [116, Theorem 4.5], $(Y_t)_{t\geq 0}$ and $(\widehat{X_t})_{t\geq 0}$ are dual processes with respect to the Revuz measure associated to A, which, by [99], is the Lebesgue measure. This completes the proof of this lemma. □

Proof of Proposition 2.2.1. The Feller property of the semigroup P is given in [14, Proposition VI.1]. Moreover, the fact that the infinitesimal generator of P is ${}^{C}D^{\alpha}_{+}$ has been proved in various papers, see e.g. [12] and [92], and the domain $\mathcal{D}_\alpha$ is given in [92, Proposition 2.2], which completes the proof of the first item. Next, from [101, Lemma 3] and its proof, we know that, up to a multiplicative positive constant, the Lebesgue measure is the unique excessive measure for P, where with the notation of [101, Example 3], $\gamma = 1 - \frac{1}{\alpha}$. Thus, since X is stochastically continuous, see [69, Lemma 2.1], a classical result from the general theory of Markov semigroups, see e.g. [38, Theorem 5.8], yields that the Feller semigroup P admits a unique extension as a contractive C_0-semigroup on $\mathrm{L}^2(\mathbb{R}_+)$. We now proceed to characterize the domain of the infinitesimal generator of the $\mathrm{L}^2(\mathbb{R}_+)$-extension, denoted by $\mathcal{D}^X$. To this end, we first observe from [11, Theorem 12.16] that since Z is a Lévy process, its semigroup $(\overline{P}_t)_{t\geq 0}$, i.e. $\overline{P}_t f(x) = \mathbb{E}_x[f(Z_t)]$, $x \in \mathbb{R}$, is a $\mathrm{L}^2(\mathbb{R})$-Markov semigroup, and its infinitesimal generator, denoted by L^Z, has the following anisotropic Sobolev space as its domain

$$\mathcal{D}^Z = \left\{ \bar{f} \in \mathrm{L}^2(\mathbb{R});\ \int_{-\infty}^{\infty} \left|\mathcal{F}_{\bar{f}}(\xi)\right|^2 |\xi|^{2\alpha} d\xi < \infty \right\}, \tag{2.12}$$

where $\mathcal{F}_{\bar{f}}(\xi) = \int_{-\infty}^{\infty} e^{i\xi x}\bar{f}(x)dx$ is the Fourier transform of $\bar{f}$. Now for a function f on $\mathbb{R}^+$ we define its extension $\bar{f} : \mathbb{R} \to \mathbb{R}$ as $\bar{f}(x) = f(x)\mathbb{I}_{\{x>0\}}$. Then, for any $f \in \overline{\mathcal{D}}_\alpha(\mathrm{L}^2(\mathbb{R}_+)) = \{f \in \mathcal{D}_\alpha(\mathrm{L}^2(\mathbb{R}_+));\ \bar{f} \in \mathrm{C}^2_0(\mathbb{R})\}$, we have clearly $\bar{f} \in \mathcal{D}^Z \cap \mathrm{C}^2_0(\mathbb{R})$ and thus by combining [14, Section I.2] and [56, Theorem 2.1] we get, that for any $x > 0$,

$$L^X f(x) = a(x) L^Z \bar{f}(x), \tag{2.13}$$

where $a(x) = \mathbb{I}_{\{x>0\}}$ from [56, (3.6)]. Therefore, since $L^Z \bar{f} \in \mathrm{L}^2(\mathbb{R})$, it is obvious that $L^X f \in \mathrm{L}^2(\mathbb{R}_+)$, which implies that $f \in \mathcal{D}^X$. Next, for any $\tau > 0$, let $f_\tau(x) = \tau^3 x^3 e^{-\tau x}, x > 0$, then easy computations yield that for all $\tau > 0$ $f_\tau \in \overline{\mathcal{D}}_\alpha(\mathrm{L}^2(\mathbb{R}_+))$, hence by the Wiener's theorem for Mellin transform $\overline{\mathcal{D}}_\alpha(\mathrm{L}^2(\mathbb{R}_+))$ is dense in $\mathrm{L}^2(\mathbb{R}_+)$ and therefore, for any $f \in \mathcal{D}_\alpha(\mathrm{L}^2(\mathbb{R}_+))$, we can take $(f_n)_{n\geq 0} \subset \overline{\mathcal{D}}_\alpha(\mathrm{L}^2(\mathbb{R}_+)) \cap \mathrm{C}_0^2(\mathbb{R}_+)$ such that $f_n \to f$ in $\mathrm{L}^2(\mathbb{R}_+)$. Writing $\bar{f}_n$ and $\bar{f}$ their corresponding extensions to $\mathrm{L}^2(\mathbb{R})$ as above, we still have $\bar{f}_n \to \bar{f}$ in $\mathrm{L}^2(\mathbb{R})$ and $\bar{f} \in D^Z$. Also note that for each $\xi \in \mathbb{R}$,

$$\mathcal{F}^+_{L^X f_n}(\xi) = \mathcal{F}_{L^Z \bar{f}_n}(\xi) = (-i\xi)^\alpha \mathcal{F}_{\bar{f}_n}(\xi) \to (-i\xi)^\alpha \mathcal{F}_{\bar{f}}(\xi) = \mathcal{F}_{L^Z \bar{f}}(\xi), \tag{2.14}$$

where we used [11, Theorem 12.16] for the second and last identity. Therefore $L^X f_n$ converges in $\mathrm{L}^2(\mathbb{R}_+)$ and $f \in \mathcal{D}^X$ by the closedness of infinitesimal generator. This shows that $\mathcal{D}_\alpha(\mathrm{L}^2(\mathbb{R}_+)) \subseteq \mathcal{D}^X$. On the other hand, take now $f \in \mathcal{D}^X \cap \mathrm{C}_0^2(\mathbb{R}_+)$ and let $\bar{f}$ be constructed as above. Then by [56, Theorem 2.6] and recalling that the fine support of $(A_t)_{t\geq 0}$ is $\mathbb{R}_+$, we have

$$L^Z \bar{f}(x) = \begin{cases} b(x) L^X f(x) & \text{for} \quad x \geq 0, \\ 0 & \text{for} \quad x < 0, \end{cases} \tag{2.15}$$

where, denoting $\mathbb{I}_+(x) = \mathbb{I}_{\{x>0\}}$,

$$b(x) = \lim_{t\to 0} \frac{\mathbb{E}_x[\int_0^t \mathbb{I}_{\{Z_s>0\}} ds]}{t} = \lim_{t\to 0} \frac{\int_0^t \overline{P}_s \mathbb{I}_+(x) ds}{t} = \lim_{t\to 0} \overline{P}_t \mathbb{I}_+(x) = \mathbb{I}_{\{x>0\}}$$

for each $x \in \mathbb{R}$. Therefore, we have

$$\int_{\mathbb{R}} \left(L^Z \bar{f}(x)\right)^2 dx = \int_{\mathbb{R}} \left((\mathbb{I}_{\{x<0\}} + \mathbb{I}_{\{x\geq 0\}}) L^Z \bar{f}(x)\right)^2 dx = \int_0^\infty \left(L^Z \bar{f}(x)\right)^2 dx = \int_0^\infty \left(L^X f(x)\right)^2 dx,$$

which implies that $\bar{f} \in \mathcal{D}^Z$ and $f \in \mathcal{D}_\alpha(\mathrm{L}^2(\mathbb{R}_+))$. Next, since we have proved that $\mathcal{D}_\alpha(\mathrm{L}^2(\mathbb{R}_+)) \subseteq \mathcal{D}^X$ and $\mathcal{D}_\alpha(\mathrm{L}^2(\mathbb{R}_+)) \cap \mathrm{C}_0^2(\mathbb{R}_+)$ is dense in $\mathrm{L}^2(\mathbb{R}_+)$, we have that $\mathcal{D}^X \cap$

$C_0^2(\mathbb{R}_+)$ is also dense in $L^2(\mathbb{R}_+)$. Hence the same argument as above shows that (2.15) still holds for any $f \in \mathcal{D}^X$, which further proves that $\mathcal{D}^X \subseteq \mathcal{D}_\alpha(L^2(\mathbb{R}_+))$ and completes the proof for the second argument. For the duality argument, we first observe from Lemma 2.2.1 that X and $\widehat{X}$ are dual processes with respect to the Lebesgue measure. Moreover, note that the minimal process X^0 belongs to the class of *positive $\frac{1}{\alpha}$-self-similar Markov processes* as introduced in [69], which also provides a bijection between positive self-similar processes and Lévy processes stated as follows. Let us define, for any $t \geq 0$, $\vartheta_t = \inf\{u > 0; \int_0^u (X_s^0)^{-\alpha} ds > t\}$, then the process

$$\xi_t^0 = \log X_{\vartheta_t}^0, \tag{2.16}$$

is a Lévy process killed at an independent exponential time. More specifically, by [92], the Laplace exponent of ξ^0 is

$$\psi^0(u) = \frac{\Gamma(u+1)}{\Gamma(u-\alpha+1)}, \ u > -1. \tag{2.17}$$

Note that by writing θ for the largest non-negative root of the convex function ψ^0, it is easy to check that $\theta = \alpha - 1 \in (0, 1)$. Hence by [101, Section 5], there exists a dual process of X^0, denoted by $\widehat{X^0}$, with the Lebesgue measure serving as the reference measure. Moreover, $\widehat{X^0}$ is also a positive $\frac{1}{\alpha}$-self-similar process with its corresponding Lévy process denoted by $\widehat{\xi^0}$, which is the dual of the Lévy process obtained from ξ^0 by means of Doob h-transform via the invariant function $h(x) = e^{\theta x}, x \in \mathbb{R}$. Therefore, the Laplace exponent of $\widehat{\xi^0}$ takes the form, for $u < 0$,

$$\widehat{\psi}(u) = \psi^0(-u+\theta) = \psi^0(-u+\alpha-1) = \frac{\Gamma(\alpha-u)}{\Gamma(-u)}.$$

Note that $\widehat{\xi^0}$ drifts to $-\infty$ a.s. and thus $\widehat{X^0}$ has a a.s. finite lifetime $T_0^{\widehat{X^0}} = \inf\{t > 0; \widehat{X}_t^0 \leq 0\}$. Hence by recalling that X can be viewed as the recurrent extension of X^0 that leaves 0 continuously a.s., we deduce from [101, Lemma 6] that $\widehat{X}$

can also be viewed as the recurrent extension of $\widehat{X}^0$ which leaves 0 by a jump according to the jump-in measure $Cx^{-\alpha}, x, C > 0$. The Feller property of the semigroup of such recurrent extension has been shown in [20, Proposition 3.1], while the existence of the $\mathrm{L}^2(\mathbb{R}_+)$-extension follows by the same argument than the one we developed for P. Moreover, from [11, Theorem 12.16], we deduce easily that $\mathcal{D}^{\widehat{Z}} = \mathcal{D}^Z$, hence using the same method as above, we get that $\mathcal{D}^{\widehat{X}} = \mathcal{D}^X = \mathcal{D}_\alpha(\mathrm{L}^2(\mathbb{R}_+))$. Finally, using the same arguments as in (2.14), we see that for any $f \in \mathcal{D}_\alpha(\mathrm{L}^2(\mathbb{R}_+))$,

$$\mathcal{F}^+_{L^{\widehat{X}} f}(\xi) = \mathcal{F}_{L^{\widehat{Z}} \bar{f}}(\xi) = (i\xi)^\alpha \mathcal{F}_{\bar{f}}(\xi).$$

Comparing this identity with [48, Lemma 2.1 and Theorem 2.3], we conclude that $L^{\widehat{X}} f = D^\alpha_- f$ on $\mathcal{D}_\alpha(\mathrm{L}^2(\mathbb{R}_+))$. This completes the proof. □

2.3 Intertwining relationship

We say that a linear operator Λ is a *multiplicative operator* if it admits the following representation, for any $f \in \mathrm{B}_b(\mathbb{R}_+)$,

$$\Lambda f(x) = \int_0^\infty f(xy)\lambda(y)dy,$$

for some integrable function λ. When in addition λ is the density of the law of a random variable $\mathbf{X}$, i.e. $\lambda(y) \geq 0$ and $\langle 1, \lambda \rangle = 1$, we say that Λ is a *Markov multiplicative operator*. Moreover, $\mathcal{M}_\lambda = \mathcal{M}_\Lambda = \mathcal{M}_{\mathbf{X}}$ is called a Markov multiplier where for at least $\Re(s) = 1$,

$$\mathcal{M}_\Lambda(s) = \int_0^\infty y^{s-1}\lambda(y)dy,$$

is the Mellin transform of λ. By adapting the developments in [114, 2.1.9] based on the Fourier transform, we also have that if $\int_0^\infty y^{-\frac{1}{2}}\lambda(y)dy < \infty$ then $\Lambda \in \mathbf{B}\mathrm{L}^2(\mathbb{R}_+)$

with, for any $f \in \mathrm{L}^2(\mathbb{R}_+)$,

$$\mathcal{M}_{\Lambda f}(s) = \mathcal{M}_\Lambda(1-s)\mathcal{M}_f(s). \tag{2.18}$$

Note that this latter provides that Λ is one-to-one in $\mathrm{L}^2(\mathbb{R}_+)$ if $\mathcal{M}_\Lambda(1-s) \neq 0$. We also recall from [83] that if $s \mapsto \mathcal{M}_\lambda(s)$ is defined, absolutely integrable and uniformly decays to zero along the lines of the strip $s \in \mathbb{C}_{(\underline{a},\overline{a})}$ for some $\underline{a} < \overline{a}$, then the Mellin inversion theorem applies to yield, for any $x > 0$,

$$\lambda(x) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} x^{-s}\mathcal{M}_\lambda(s)ds, \quad \underline{a} < a < \overline{a}. \tag{2.19}$$

Now we are ready to state the following.

Theorem 2.3.1. *Let us write, for any $\alpha \in (1,2)$,*

$$\mathcal{M}_{\Lambda_\alpha}(s) = \frac{\Gamma(\frac{s-1}{\alpha}+1)\Gamma(\frac{s}{\alpha})}{\Gamma(\frac{1}{\alpha})\Gamma(s)}, \quad s \in \mathbb{C}_+. \tag{2.20}$$

Then, the following holds.

1. *$\mathcal{M}_{\Lambda_\alpha}$ is a Markov multiplier and $\Lambda_\alpha \in \mathbf{B}(\mathrm{L}^2(\mathbb{R}_+)) \cap \mathbf{B}(\mathrm{C}_0(\mathbb{R}_+))$. Moreover, it is one-to-one on $\mathrm{C}_0(\mathbb{R}_+)$, and, in $\mathrm{L}^2(\mathbb{R}_+)$, $\overline{\mathrm{Ran}}(\Lambda_\alpha) = \mathrm{L}^2(\mathbb{R}_+)$.*
2. *Moreover, for any $t \geq 0$ and $f \in \mathrm{L}^2(\mathbb{R}_+)$, the following intertwining relation holds*

$$P_t\Lambda_\alpha f = \Lambda_\alpha Q_t f, \tag{2.21}$$

   where $Q = (Q_t)_{t\geq 0}$ is the $\mathrm{L}^2(\mathbb{R}_+)$-extension of the α-Bessel self-adjoint semigroup as defined in Appendix 2.7.
3. *Consequently, we have, for any $f \in \mathcal{D}_{\mathbf{L}}(\mathrm{L}^2(\mathbb{R}_+))$,*

$$^C D_+^\alpha \Lambda_\alpha f = \Lambda_\alpha \mathbf{L} f, \tag{2.22}$$

   where the fractional operator $^C D_+^\alpha$ was defined in (2.2), while the second order differential operator $\mathbf{L}$ and its $\mathrm{L}^2(\mathbb{R}_+)$-domain $\mathcal{D}_{\mathbf{L}}(\mathrm{L}^2(\mathbb{R}_+))$ are defined in (2.61) and (2.67), respectively.

The proof of this Theorem is split into three steps. First, we show that (2.20) is indeed a Markov multiplier. Then, we establish the identity (2.21) in the space $C_0(\mathbb{R}_+)$. Finally, by remarking that $C_0(\mathbb{R}_+)$ is dense in $L^2(\mathbb{R}_+)$, we can extend the intertwining identity to $L^2(\mathbb{R}_+)$ by a continuity argument.

2.3.1 The Markov multiplicative operator Λ_α

In order to prove Theorem 2.3.1(1), which provides some substantial properties of Λ_α, we shall need the following claims. Here and throughout the rest of this section, we set, for any $\alpha, \tau > 0$,

$$\mathbf{e}_{\alpha,\tau}(x) = \mathrm{d}_{\tau^{\frac{1}{\alpha}}} \mathbf{e}_\alpha (x) = e^{-\tau x^\alpha}, \quad x > 0. \tag{2.23}$$

Lemma 2.3.1. *Let us define*

$$g_\alpha(z) \quad = \quad \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{1}{\alpha}\right)\Gamma(\alpha n + 1)}{\Gamma(n + \frac{1}{\alpha})(n!)^2}(e^{i\pi} z^\alpha)^n, \tag{2.24}$$

then g_α is holomorphic on $\mathbb{C}_{(-\infty,0)^c}$. Moreover, $g_\alpha \in L^2(\mathbb{R}_+)$ with $\Lambda_\alpha g_\alpha = \mathbf{e}_\alpha$ where $\mathbf{e}_\alpha$ is defined in (2.23).

Proof. First, from the Stirling approximation

$$\Gamma(a) \overset{\infty}{\sim} \sqrt{2\pi} a^{a-\frac{1}{2}} e^{-a}, \tag{2.25}$$

see [70, (1.4.25)], we get that $\frac{\Gamma(\alpha n+\alpha+1)\Gamma(n+\frac{1}{\alpha})}{\Gamma(\alpha n+1)\Gamma(n+1+\frac{1}{\alpha})(n+1)^2} \overset{\infty}{=} O(n^{\alpha-3})$, hence, as $\alpha \in (1, 2)$, g_α is holomorphic on $\mathbb{C}_{(-\infty,0)^c}$. We now proceed to show that $g_\alpha \in L^2(\mathbb{R}_+)$. To this end, let us define, for $0 < \Re(s) < 1$,

$$\mathcal{M}_\alpha (s) = \frac{\Gamma(1 + \frac{1}{\alpha})\Gamma(\frac{s}{\alpha})\Gamma(1 - s)}{\Gamma(1 - \frac{s}{\alpha})\Gamma(\frac{1-s}{\alpha})}$$

and we first aim at proving that $\mathcal{M}_\alpha = \mathcal{M}_{g_\alpha}$ the Mellin transform of g_α. For this purpose, observe that $s \mapsto \mathcal{M}_\alpha(s)$ is holomorphic on $\mathbb{C}_{(0,1)}$ and then consider the contour integral $I_{N,B} = \frac{1}{2\pi i}\int_{C_{N,B}} z^{-s}\mathcal{M}_\alpha(s)\,ds$ where $C_{N,B}$ is the rectangle with vertices at $\frac{1}{2} \pm iB$ and $-\alpha N - \frac{\alpha}{2} \pm iB$ for some large $N \in \mathbb{N}$ and $B > 0$. Then we can obviously split $I_{N,B}$ into four parts, namely $I_{N,B} = I_1 + I_2 + I_3 + I_4$ where

$$I_1 = \frac{1}{2\pi i}\int_{\frac{1}{2}+iB}^{-\alpha N-\frac{\alpha}{2}+iB} z^{-s}\mathcal{M}_\alpha(s)ds, \quad I_2 = \frac{1}{2\pi i}\int_{-\alpha N-\frac{\alpha}{2}+iB}^{-\alpha N-\frac{\alpha}{2}-iB} z^{-s}\mathcal{M}_\alpha(s)ds,$$
$$I_3 = \frac{1}{2\pi i}\int_{-\alpha N-\frac{\alpha}{2}-iB}^{\frac{1}{2}-iB} z^{-s}\mathcal{M}_\alpha(s)ds, \quad I_4 = \frac{1}{2\pi i}\int_{\frac{1}{2}-iB}^{\frac{1}{2}+iB} z^{-s}\mathcal{M}_\alpha(s)ds.$$

Next, observing from the Stirling approximation, see e.g. [83, (2.1.8)], that for fixed $a \in \mathbb{R}$,

$$|\Gamma(a+ib)| \overset{\pm\infty}{\sim} C|b|^{a-\frac{1}{2}}e^{-\frac{\pi}{2}|b|}, \tag{2.26}$$

with $C = C(a) > 0$, we deduce, for some $C_\alpha > 0$, that

$$|\mathcal{M}_\alpha(a+ib)| \overset{\pm\infty}{\sim} C_\alpha|b|^{\frac{3}{\alpha}a-a-\frac{1}{\alpha}}e^{-\frac{\pi}{2}(1-\frac{1}{\alpha})|b|}, \tag{2.27}$$

and, hence

$$\left|z^{-(a+ib)}\mathcal{M}_\alpha(a+ib)\right| \overset{\pm\infty}{\sim} C_\alpha|z|^{-a}|b|^{\frac{3}{\alpha}a-a-\frac{1}{\alpha}}e^{-\frac{\pi}{2}(1-\frac{1}{\alpha})|b|+\arg(z)b}. \tag{2.28}$$

Therefore, if $|\arg(z)| < \frac{\pi}{2}(1-\frac{1}{\alpha})$ and N is kept fixed, we have both

$$\lim_{B\to\infty}|I_1| = \lim_{B\to\infty}|I_3| = 0. \tag{2.29}$$

For the integral I_2, we have

$$\begin{aligned}|I_2| &\le \frac{1}{2\pi}|z|^{\alpha N+\frac{\alpha}{2}}\int_{-\infty}^{\infty} e^{\arg(z)b}\left|\frac{\Gamma(1+\frac{1}{\alpha})\Gamma(-N-\frac{1}{2}+i\frac{b}{\alpha})\Gamma(1+\alpha N+\frac{\alpha}{2}+ib)}{\Gamma(N+\frac{3}{2}-i\frac{b}{\alpha})\Gamma(N+\frac{1}{2}+\frac{1}{\alpha}-i\frac{b}{\alpha})}\right|db\\
&= \frac{1}{2}|z|^{\alpha N+\frac{\alpha}{2}}\int_{-\infty}^{\infty} e^{\arg(z)b}\left|\frac{\Gamma(1+\frac{1}{\alpha})\Gamma(1+\alpha N+\frac{\alpha}{2}+ib)}{\Gamma(N+\frac{3}{2}-i\frac{b}{\alpha})^2\Gamma(N+\frac{1}{2}+\frac{1}{\alpha}-i\frac{b}{\alpha})\cosh(\frac{\pi b}{\alpha})}\right|db\end{aligned}$$

where we have used the reflection formula for the gamma function. Using the Stirling approximation again, it is easy to derive, for large N, the upper bound

$$\left|\frac{\Gamma(1+\alpha N+\frac{\alpha}{2}+ib)}{\Gamma(N+\frac{3}{2}-i\frac{b}{\alpha})^2}\right| \le Ce^{N(\alpha\log\alpha-\alpha-2)}N^{(\alpha-2)N+\frac{\alpha-1}{2}}$$

which is uniform in $b \in \mathbb{R}$ and where $C > 0$. Moreover, recalling, from [83, (5.1.3)], that, for $N \geq 1$ and $b \in \mathbb{R}$, $\left|\Gamma(N + \frac{1}{2} + \frac{1}{\alpha} - i\frac{b}{\alpha})\right| \geq \frac{\Gamma(N+\frac{1}{2}+\frac{1}{\alpha})}{\cosh^{\frac{1}{2}}(\frac{\pi b}{\alpha})}$, we find

$$|I_2| \leq C \frac{e^{N(\alpha \log \alpha - \alpha - 2)} N^{(\alpha-2)N + \frac{\alpha-1}{2}}}{\Gamma(N + \frac{1}{2} + \frac{1}{\alpha})} \int_{-\infty}^{\infty} \frac{e^{\arg(z) b}}{\cosh^{\frac{1}{2}}(\frac{\pi b}{\alpha})} db$$

where the last integral converges absolutely whenever $|\arg(z)| < \frac{\pi}{2\alpha}$. For such z, since $1 < \alpha < 2$, we get that $\lim_{N\to\infty} |I_2| = 0$. Therefore, combining this with (2.29), we have, for $|\arg(z)| < \frac{\pi}{2\alpha} \wedge \frac{\pi}{2}(1 - \frac{1}{\alpha}) = \frac{\pi}{2\alpha}$,

$$\lim_{N,B\to\infty} I_{N,B} = \lim_{B\to\infty} I_4 = \frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} z^{-s} \mathcal{M}_\alpha(s)\, ds.$$

Hence an application of Cauchy's integral theorem yields

$$\frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} z^{-s} \mathcal{M}_\alpha(s)\, ds = \sum_{n=0}^{\infty} \frac{\Gamma(\frac{1}{\alpha})\Gamma(\alpha n + 1)}{\Gamma(n + \frac{1}{\alpha})(n!)^2} (-1)^n z^{\alpha n} = g_\alpha(z) \tag{2.30}$$

where we sum over the poles $s = -\alpha n, n = 0, 1 \ldots$ of $\Gamma\left(\frac{s}{\alpha}\right)$ with residues $\frac{\alpha(-1)^n}{n!}$. This shows that $\mathcal{M}_{g_\alpha} = \mathcal{M}_\alpha$. Since $\alpha \in (1, 2)$, we have, from (2.27), that $b \mapsto \mathcal{M}_\alpha(\frac{1}{2} + ib) \in \mathrm{L}^2(\mathbb{R})$ and by the Parseval identity for the Mellin transform we conclude that $g_\alpha \in \mathrm{L}^2(\mathbb{R}_+)$. Finally, by means of a standard application of Fubini theorem, see e.g. [113, Section 1.77]), one shows that, for any $x > 0$,

$$\Lambda_\alpha g_\alpha(x) = \sum_{n=0}^{\infty} \frac{\Gamma(\frac{1}{\alpha})\Gamma(\alpha n + 1)}{\Gamma(n + \frac{1}{\alpha})(n!)^2} \mathcal{M}_{\Lambda_\alpha}(\alpha n + 1)(-1)^n x^{\alpha n} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{\alpha n}}{n!} = \mathbf{e}_\alpha(x),$$

where we used the expression (2.20). This completes the proof of the lemma. □

Next, let us show that $\mathcal{M}_{\Lambda_\alpha}$ is the Mellin transform of a random variable that we denote by I_α. To this end, we write, for any $u > 0$,

$$\phi_\alpha(u) = \frac{\Gamma(\alpha u + 1)}{\Gamma(\alpha u + 1 - \alpha)} \frac{1}{u - 1 + \frac{1}{\alpha}} = \frac{\alpha}{\Gamma(2 - \alpha)} + \int_0^\infty (1 - e^{-uy}) \frac{\alpha(\alpha - 1)}{\Gamma(2 - \alpha)} \frac{e^{-\frac{y}{\alpha}}}{(1 - e^{-\frac{y}{\alpha}})^\alpha} dy,$$

where the second identity follows after some standard computation, see e.g. [86, (4.2)]. As plainly $\int_0^\infty (y \wedge 1) \frac{e^{-\frac{y}{\alpha}}}{(1-e^{-\frac{y}{\alpha}})^\alpha} dy < \infty$, we get, from [107, Theorem 3.2], that ϕ_α

is a Bernstein function, whose definition is given in [107, Definition 3.2]. Moreover, by [107, Section 5], ϕ_α is the Laplace exponent of a subordinator, that is an increasing process with stationary and independent increments, which we denote by $(\xi_t)_{t\geq 0}$. Next, observing that for any $n \in \mathbb{N}$,

$$\mathcal{M}_{\Lambda_\alpha}(\alpha n + 1) = \frac{n!\Gamma(n+\frac{1}{\alpha})}{\Gamma(\frac{1}{\alpha})\Gamma(\alpha n + 1)} = \frac{n!}{\prod_{k=1}^n \phi_\alpha(k)}, \tag{2.31}$$

we deduce, from [30, Proposition 3.3], that $(\mathcal{M}_{\Lambda_\alpha}(\alpha n + 1))_{n\geq 0}$ is the Stieltjes moment sequence of the random variable $\int_0^\infty e^{-\xi_t}dt$. Moreover, observe from its definition (2.20) and applications of the recurrence relation of the gamma function that $\mathcal{M}_{\Lambda_\alpha}$ satisfies the functional equation, on $s \in \mathbb{C}_+$,

$$\mathcal{M}_{\Lambda_\alpha}(\alpha s + 1) = \frac{s}{\phi_\alpha(s)}\mathcal{M}_{\Lambda_\alpha}(\alpha(s-1)+1), \quad \mathcal{M}_{\Lambda_\alpha}(1) = 1,$$

hence, by a uniqueness argument developed in [91, Section 7], we have

$$\mathcal{M}_{\Lambda_\alpha}(s+1) = \mathbb{E}\left[\left(\int_0^\infty e^{-\xi_t}dt\right)^{\frac{s}{\alpha}}\right] = \mathbb{E}\left[I_\alpha^s\right]. \tag{2.32}$$

Consequently $\mathcal{M}_{\Lambda_\alpha}(s)$ is the Mellin transform of the variable $I_\alpha = \left(\int_0^\infty e^{-\xi_s}ds\right)^{\frac{1}{\alpha}}$. Finally, since the law of $\int_0^\infty e^{-\xi_t}dt$ is known to be absolutely continuous, see e.g. [91, Proposition 7.7], so is the one of I_α, therefore we conclude that $\mathcal{M}_{\Lambda_\alpha}$ is indeed a Markov multiplier, which provides the first claim of Theorem 2.3.1(1). Next, the one-to-one property of Λ_α follows from the fact that the mapping $s \mapsto \mathcal{M}_{\Lambda_\alpha}(s)$ is clearly zero-free on the line $1 + i\mathbb{R}$. Moreover, writing λ_α the density of I_α, we have by dominated convergence that for any $f \in \mathrm{C}_0(\mathbb{R}_+)$, $\Lambda_\alpha f \in \mathrm{C}_0(\mathbb{R}_+)$ with $\|\Lambda_\alpha f\|_\infty \leq \|f\|_\infty$, that is, $\Lambda_\alpha \in \mathbf{B}(\mathrm{C}_0(\mathbb{R}_+))$. On the other hand, for $f \in \mathrm{L}^2(\mathbb{R}_+)$, Jensen's inequality and a change of variable yield

$$\|\Lambda_\alpha f\|^2 = \int_0^\infty \mathbb{E}\left[f(xI_\alpha)\right]^2 dx \leq \int_0^\infty \mathbb{E}\left[f^2(xI_\alpha)\right]dx = \mathbb{E}[I_\alpha^{-1}]\,\|f\|^2$$

where $\mathbb{E}[I_\alpha^{-1}] = \mathcal{M}_{\Lambda_\alpha}(0) = \frac{\Gamma(1-\frac{1}{\alpha})}{\Gamma(1+\frac{1}{\alpha})} < \infty$. Hence $\Lambda_\alpha \in \mathbf{B}(\mathrm{L}^2(\mathbb{R}_+))$. Moreover, from Lemma 2.3.1, it is easy to conclude that $\Lambda_\alpha \mathrm{d}_q g_\alpha = \mathrm{d}_q \mathbf{e}_\alpha$ for all $q > 0$, where

$d_q g_\alpha \in L^2(\mathbb{R}_+)$ since $q d_q$ is a unitary operator in $L^2(\mathbb{R}_+)$. Hence, by the well-known result that $\overline{\mathrm{Span}}(d_q \mathbf{e}_\alpha)_{q>0} = L^2(\mathbb{R}_+)$, we have that Λ_α has a dense range in $L^2(\mathbb{R}_+)$, which completes the proof of Theorem 2.3.1(1).

2.3.2 Proofs of Theorem 2.3.1(2) and (3)

We recall that a collection of σ-finite measures $(\eta_t)_{t>0}$ is called an *entrance law* for the semigroup P if for any $t, s > 0$ and any $f \in C_0(\mathbb{R}_+), \eta_t P_s f = \eta_{t+s} f$ where $\eta_t f = \int_0^\infty f(x)\eta_t(dx)$. We also recall from Appendix 2.7 that G_α is the $\frac{1}{\alpha}$ power of a gamma variable with parameter $\frac{1}{\alpha} > 0$, that is $\mathbb{P}(G_\alpha \in dy) = \frac{e^{-y^\alpha}}{\Gamma(1+\frac{1}{\alpha})} dy, y > 0$. Now we are ready to state the following Lemma.

Lemma 2.3.2. *P admits an entrance law $(\eta_t)_{t>0}$ defined for any $t > 0$ by $\eta_t f = \eta_1 d_t f = \int_0^\infty f(ty)\eta_1(dy)$ where $\eta_1(dy) = \lambda_{\mathbf{X}_\alpha}(y)dy$, with $\lambda_{\mathbf{X}_\alpha} \in L^2(\mathbb{R}_+)$, is the probability measure of a variable $\mathbf{X}_\alpha$. Its Mellin transform takes the form*

$$\mathcal{M}_{\mathbf{X}_\alpha}(s) = \frac{\Gamma(s)}{\Gamma(\frac{s}{\alpha} + 1 - \frac{1}{\alpha})}, \quad s \in \mathbb{C}_+. \tag{2.33}$$

Moreover, we have the following factorization of the variable G_α

$$G_\alpha \stackrel{d}{=} \mathbf{X}_\alpha \times I_\alpha,$$

where $\stackrel{d}{=}$ stands for the identity in distribution and $\mathbf{X}_\alpha$ is considered independent of I_α, which we recall was characterized in (2.32).

Proof. First, let us observe from (2.33) that for any $n \geq 0$,

$$\mathcal{M}_{\mathbf{X}_\alpha}(\alpha n + 1) = \frac{\Gamma(\alpha n + 1)}{n!} = \frac{\prod_{k=1}^n \frac{\Gamma(\alpha k+1)}{\Gamma(\alpha(k-1)+1)}}{n!} = \prod_{k=1}^n \frac{\psi^0(\alpha k)}{k}, \tag{2.34}$$

where we recall from the proof of Proposition 2.2.1 that $\psi^0(u) = \frac{\Gamma(u+1)}{\Gamma(u-\alpha+1)}, u > \alpha - 1$, is the Laplace exponent of the killed Lévy process ξ^0 defined in (2.16). Then by [8, Theorem 1], we deduce that $(\mathcal{M}_{\mathbf{X}_\alpha}(\alpha n+1))_{n\geq 0}$ is the moment sequence of the variable X_1^α under $\mathbb{P}_0$, for which we used the fact that since X is a $\frac{1}{\alpha}$-self-similar process, X^α is a 1-self-similar process whose minimal process is associated, through the Lamperti mapping, to a Lévy process with Laplace exponent $\psi_\alpha(u) = \psi^0(\alpha u)$. Moreover, note from (2.33) that $\mathcal{M}_{\mathbf{X}_\alpha}$ satisfies the functional equation $\mathcal{M}_{\mathbf{X}_\alpha}(\alpha s+1) = \frac{\psi_\alpha(s)}{s}\mathcal{M}_{\mathbf{X}_\alpha}(\alpha(s-1)+1)$ with $\mathcal{M}_{\mathbf{X}_\alpha}(1) = 1$, hence by a uniqueness argument, see again [91, Section 7], we conclude that $\mathcal{M}_{\mathbf{X}_\alpha}(s+1) = \mathbb{E}_0[X_1^s]$ is indeed the Mellin transform of X_1 under $\mathbb{P}_0$. Using again the Stirling approximation (2.26), we see that $|\mathcal{M}_{\mathbf{X}_\alpha}(\frac{1}{2}+ib)| \overset{\pm\infty}{=} \mathrm{O}\left(|b|^{\frac{1}{2\alpha}-\frac{1}{2}}e^{-\frac{\pi}{2}(1-\frac{1}{\alpha})|b|}\right)$, and thus $b \mapsto \mathcal{M}_{\mathbf{X}_\alpha}(\frac{1}{2}+ib) \in \mathrm{L}^2(\mathbb{R})$. Hence by Mellin inversion and Parseval identity, we get that the law of $\mathbf{X}_\alpha$ is absolutely continuous with a density $\lambda_{\mathbf{X}_\alpha} \in \mathrm{L}^2(\mathbb{R}_+)$. Now, recalling that $\eta_1(dy) = \lambda_{\mathbf{X}_\alpha}(y)dy$ and for any $t>0$, $\eta_t f = \eta_1 \mathrm{d}_t f$, we get, from (2.34) augmented by a moment identification that $(\eta_t)_{t>0}$ is an entrance law for the semigroup P. Finally, from the expression of $\mathcal{M}_{\Lambda_\alpha}$ in (2.20), we conclude that for $s \in \mathbb{C}_+$,

$$\mathcal{M}_{\mathbf{X}_\alpha}(s)\mathcal{M}_{\Lambda_\alpha}(s) = \frac{\Gamma(s)}{\Gamma(\frac{s}{\alpha}+1-\frac{1}{\alpha})}\frac{\Gamma(\frac{s-1}{\alpha}+1)\Gamma(\frac{s}{\alpha})}{\Gamma(\frac{1}{\alpha})\Gamma(s)} = \frac{\Gamma(\frac{s}{\alpha})}{\Gamma(\frac{1}{\alpha})} = \mathcal{M}_{G_\alpha}(s)$$

where we used for the last identity the expression (2.63). We complete the proof by invoking the injectivity of the Mellin transform. □

We are now ready to prove the intertwining relation stated in Theorem 2.3.1(2). First, since $s \mapsto \mathcal{M}_{\mathbf{X}_\alpha}(s)$ is zero-free on the line $1+i\mathbb{R}$, we again conclude that the Markov operator $\Lambda_{\mathbf{X}_\alpha}$ associated to the positive variable $\mathbf{X}_\alpha$, i.e. $\Lambda_{\mathbf{X}_\alpha}f(x) = \int_0^\infty f(xy)\lambda_{\mathbf{X}_\alpha}(y)dy$, is injective on $\mathrm{C}_0(\mathbb{R}_+)$. This combined with the fact that the law of G_α is the entrance law at time 1 of the semigroup Q and with the factorization of this latter stated in Lemma 2.3.2 provide all conditions for the application of [30, Proposition 3.2], which gives that for any $t \geq 0$ and

$f \in \mathrm{C}_0(\mathbb{R}_+)$, the following intertwining relationship between the Feller semi-groups $(P_t^F)_{t\geq 0}$ and $(Q_t^F)_{t\geq 0}$,

$$P_t^F \Lambda_\alpha f = \Lambda_\alpha Q_t^F f. \tag{2.35}$$

in $\mathrm{C}_0(\mathbb{R}_+)$. Futhermore, since $\mathrm{C}_0(\mathbb{R}_+) \cap \mathrm{L}^2(\mathbb{R}_+)$ is dense in $\mathrm{L}^2(\mathbb{R}_+)$, we can extend the intertwining identity into $\mathrm{L}^2(\mathbb{R}_+)$ by continuity of the involved operators and complete the proof of Theorem 2.3.1(2). Finally, Theorem 2.3.1(3) follows directly from (2.21) by recalling that ${}^C D_+^\alpha$ and $\mathbf{L}$ are the infinitesimal generators of P and Q, respectively, where the $\mathrm{L}^2(\mathbb{R}_+)$-domain of $\mathbf{L}$ is given in (2.67). This concludes the proof of Theorem 2.3.1.

2.4 Eigenfunctions and upper frames

We start by recalling a few definitions concerning the spectrum of linear operators and we refer to [46, XV.8] for a thorough account on these objects. Let $\mathrm{P} \in \mathbf{B}(\mathrm{L}^2(\mathbb{R}_+))$. We say that a complex number $\mathfrak{z} \in \sigma(\mathrm{P})$, the spectrum of P, if $\mathrm{P} - \mathfrak{z}I$ does not have an inverse in $\mathrm{L}^2(\mathbb{R}_+)$ Moreover, we also recall from [2] that a collection of functions $(g_q)_{q>0}$ is a *frame* for $\mathrm{L}^2(\mathbb{R}_+)$ if for all $q > 0$ $g_q \in \mathrm{L}^2(\mathbb{R}_+)$ and there exists constants $A, B > 0$, called the *frame bounds*, such that, for all $f \in \mathrm{L}^2(\mathbb{R}_+)$,

$$A||f||^2 \leq \int_0^\infty \left\langle f, g_q \right\rangle^2 dq \leq B||f||^2.$$

Moreover, we say $(g_q)_{q>0}$ is *upper frame* if it only satisfies the second inequality. Finally, recalling that $\mathcal{J}_\alpha$ was defined in (2.7), we are ready to state the following claims which include the expression along with substantial properties of the set of eigenfunctions of P_t.

Theorem 2.4.1. *1. For any $q, t > 0$, $\mathrm{d}_q\mathcal{J}_\alpha$ is an eigenfunction for P_t associated to the eigenvalue $e^{-q^\alpha t}$. Consequently, we have $(e^{-q^\alpha t})_{q>0} \subseteq S_p(P_t)$.*

2. Let the linear operator $\mathcal{H}_\alpha$ be defined for any $f \in \mathrm{L}^2(\mathbb{R}_+)$ by

$$\mathcal{H}_\alpha f(q) = \int_0^\infty f(x)\mathcal{J}_\alpha(qx)dx, \quad q > 0, \tag{2.36}$$

then $\mathcal{H}_\alpha \in \mathbf{B}(\mathrm{L}^2(\mathbb{R}_+))$ with $|||\mathcal{H}_\alpha||| = \sup_{\|f\|=1} \|\mathcal{H}_\alpha f\| \le \frac{\Gamma(1-\frac{1}{\alpha})}{\Gamma(1+\frac{1}{\alpha})}$. Consequently, the collection of functions $(\mathrm{d}_q\mathcal{J}_\alpha)_{q\ge 0}$ is a dense upper frame for $\mathrm{L}^2(\mathbb{R}_+)$, with upper frame bound $\frac{\Gamma(1-\frac{1}{\alpha})}{\Gamma(1+\frac{1}{\alpha})}$.

3. For any $k \in \mathbb{N}$, $\mathcal{J}_\alpha^{(k)}$ admits the following asymptotic expansion for large $x > 0$,

$$\mathcal{J}_\alpha^{(k)}(x) \quad \approx \quad \frac{x^{-k-\alpha}}{\pi\Gamma(1+\frac{1}{\alpha})} \sum_{n=0}^{\infty} a_{n,k}\, x^{-\alpha n} \tag{2.37}$$

where $a_{n,k} = (-1)^{n+k}\Gamma(\alpha n + \alpha + k)\sin(\pi\alpha(n+1))$ and $\approx$ means that for any $N \in \mathbb{N}$, $\mathcal{J}_\alpha^{(k)}(x) - \frac{x^{-k-\alpha}}{\pi\Gamma(1+\frac{1}{\alpha})}\sum_{n=0}^{N} a_{n,k}\, x^{-\alpha n} \stackrel{\infty}{=} \mathrm{o}\left(x^{-k-\alpha-\alpha(N+1)}\right)$.

Remark 2.4.1. *Note that there is a cut-off in the nature of the spectrum when one considers the family of operators P indexed by $\alpha \in (1,2)$. Indeed, when $\alpha = 2$, then $\mathcal{J}_2 = J_2 \notin \mathrm{L}^2(\mathbb{R}_+)$ (see (2.64) for the definition of the Bessel-type function J_2) and hence, for all $q, t > 0$, $e^{-q^2 t} \notin S_p(P_t)$ but instead $e^{-q^2 t} \in S_c(P_t)$.*

Proof. First, we recall that J_α, the Bessel-type function, is defined in (2.64) as an holomorphic function on $\mathbb{C}_{(-\infty,0)^c}$. As $\alpha \in (1,2)$ and $J_\alpha(x) \stackrel{\infty}{=} \mathrm{O}\left(x^{\frac{\alpha-2}{4}}\right)$, see e.g. [109], we get that $J_\alpha \in \mathrm{C}_0(\mathbb{R}_+)$. Hence, as above, applying Fubini's theorem, we obtain, for $x > 0$, that

$$\Lambda_\alpha J_\alpha(x) = \alpha \sum_{n=0}^{\infty} \frac{(e^{i\pi}x^\alpha)^n}{n!\Gamma(n+\frac{1}{\alpha})}\mathcal{M}_{\Lambda_\alpha}(\alpha n + 1) = \frac{1}{\Gamma(1+\frac{1}{\alpha})}\sum_{n=0}^{\infty}\frac{(e^{i\pi}x^\alpha)^n}{\Gamma(\alpha n+1)} = \mathcal{J}_\alpha(x), \tag{2.38}$$

which shows, since $\Lambda_\alpha \in \mathbf{B}(\mathrm{C}_0(\mathbb{R}_+))$, that both $\mathcal{J}_\alpha \in \mathrm{C}_0(\mathbb{R}_+)$ and $\mathrm{d}_q\mathcal{J}_\alpha \in \mathrm{C}_0(\mathbb{R}_+)$ for all $q > 0$. Thus, we can use the relation (2.35) to get, for all $q > 0$ and $x \ge 0$,

$$P_t^F \mathrm{d}_q\mathcal{J}_\alpha(x) = P_t^F \Lambda_\alpha \mathrm{d}_q J_\alpha(x) = \Lambda_\alpha Q_t^F \mathrm{d}_q J_\alpha(x) = e^{-q^\alpha t}\Lambda_\alpha \mathrm{d}_q J_\alpha(x) = e^{-q^\alpha t}\mathrm{d}_q\mathcal{J}_\alpha(x). \tag{2.39}$$

Next, proceeding as in the proof of Lemma 2.3.1, we get, for $|\arg(z)| < \left(\frac{1}{\alpha} - \frac{1}{2}\right)\pi$, that

$$\mathcal{J}_\alpha(z) = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} z^{-s}\mathcal{M}_{\mathcal{J}_\alpha}(s)ds,$$

where, for $0 < \Re(s) < \alpha$,

$$\mathcal{M}_{\mathcal{J}_\alpha}(s) = \frac{\Gamma(1-\frac{s}{\alpha})\Gamma(\frac{s}{\alpha})}{\Gamma(1-s)\Gamma(\frac{1}{\alpha})}. \tag{2.40}$$

Since from (2.26), $\left|\mathcal{M}_{\mathcal{J}_\alpha}\left(\frac{1}{2}+ib\right)\right| \stackrel{\pm\infty}{=} \mathrm{O}\left(|b|^{\alpha a-\frac{\alpha}{2}}e^{-\frac{\pi}{2}(\frac{2}{\alpha}-1)|b|}\right)$, we get, by the Parseval identity, that $\mathcal{J}_\alpha \in \mathrm{L}^2(\mathbb{R}_+)$. Moreover, since P^F coincides with its extension P on $\mathrm{C}_0(\mathbb{R}_+)\cap\mathrm{L}^2(\mathbb{R}_+)$, we conclude from (2.39) that, for all $q,t>0$, $\mathrm{d}_q\mathcal{J}_\alpha$ is an eigenfunction of P_t with eigenvalue $e^{-q^\alpha t}$. This completes the proof of the first item. Next, with $\widehat{\Lambda}_\alpha \in \mathbf{B}(\mathrm{L}^2(\mathbb{R}_+))$ as the $\mathrm{L}^2(\mathbb{R}_+)$-adjoint of $\Lambda_\alpha \in \mathbf{B}(\mathrm{L}^2(\mathbb{R}_+))$, we have, for any $f\in\mathrm{L}^2(\mathbb{R}_+)$,

$$||\mathcal{H}_\alpha f||^2 = \int_0^\infty \left\langle f, \Lambda_\alpha \mathrm{d}_q J_\alpha\right\rangle^2 dq = ||H_\alpha\widehat{\Lambda}_\alpha f||^2 = ||\widehat{\Lambda}_\alpha f||^2 \leq \frac{\Gamma^2(1-\frac{1}{\alpha})}{\Gamma^2(1+\frac{1}{\alpha})}||f||^2$$

where we used successively the identity (2.38), the definition as well as the unitary property of H_α, see Proposition 2.7.1, and the identity $|||\widehat{\Lambda}_\alpha||| = |||\Lambda_\alpha||| \leq \frac{\Gamma(1-\frac{1}{\alpha})}{\Gamma(1+\frac{1}{\alpha})}$, giving the second claim. Furthermore, for any $k\in\mathbb{N}$, by a classical argument since the first series below is easily checked to be uniformly convergent in $z\in\mathbb{C}_{(-\infty,0)^c}$, we get

$$z^k\Gamma(1+\frac{1}{\alpha})\mathcal{J}_\alpha^{(k)}(z) = \sum_{n=0}^\infty \frac{(e^{i\pi}z^\alpha)^n}{\Gamma(\alpha n-k+1)} = {}_1\Psi_1(e^{i\pi}z^\alpha) \approx z^{-\alpha}\sum_{n=0}^\infty \frac{(-1)^n}{n!}\frac{\Gamma(n+1)\,z^{-\alpha n}}{\Gamma(-\alpha n-\alpha-k+1)},$$

where, for $|\arg(z)| < \frac{\pi}{2}(2-\alpha)$, we used [82, Theorem 1], with the notation therein, that is, ${}_1\Psi_1$ stands for the Wright function and we made the choice of parameters $p=q=1, \alpha_1=1, a_1=1, \beta_1=\alpha, b_1=-k+1, \kappa=1+\beta_1-\alpha_1=\alpha\in(1,2)$. The proof is completed by an application of the reflection formula for the gamma function. $\square$

2.5 Co-residual functions

In this section, we focus on characterizing the spectrum of the $L^2(\mathbb{R}_+)$-adjoint operator $\widehat{P} = (\widehat{P}_t)_{t\geq 0}$. We point out that the non-self-adjointness of P_t does not ensure the existence of eigenfunctions for $\widehat{P}_t$. In fact, we shall show in the following Lemma that the point spectrum of $\widehat{P}_t$ is empty and $S_p(P_t) = S_r(\widehat{P}_t)$, the residual spectrum of $\widehat{P}_t$.

Lemma 2.5.1. *For each $t \geq 0$, $S_p(\widehat{P}_t) = \emptyset$ and $(e^{-q^\alpha t})_{q>0} \subseteq S_r(\widehat{P}_t) = S_p(P_t)$.*

Proof. Assume that there exists $\mathfrak{z} \in S_p(\widehat{P}_t)$, then there exists a non-zero function $f_\mathfrak{z} \in L^2(\mathbb{R}_+)$ such that $\widehat{P}_t f_\mathfrak{z} = \mathfrak{z} f_\mathfrak{z}$. Moreover, since Λ_α has a dense range in $L^2(\mathbb{R}_+)$, we see that $\mathrm{Ker}(\widehat{\Lambda}_\alpha) = \{0\}$ and therefore $g_\mathfrak{z} = \widehat{\Lambda}_\alpha f_\mathfrak{z} \neq 0$ with $g_\mathfrak{z} \in L^2(\mathbb{R}_+)$ as $\widehat{\Lambda}_\alpha \in \mathbf{B}L^2(\mathbb{R}_+)$. Now by the adjoint intertwining relation of (2.21), we have

$$Q_t g_\mathfrak{z} = Q_t \widehat{\Lambda}_\alpha f_\mathfrak{z} = \widehat{\Lambda}_\alpha \widehat{P}_t f_\mathfrak{z} = \mathfrak{z} \widehat{\Lambda}_\alpha f_\mathfrak{z} = \mathfrak{z} g_\mathfrak{z},$$

which implies that $\mathfrak{z} \in S_p(Q_t)$, a contradiction to the fact that $S_p(Q_t) = \emptyset$. Therefore we have $S_p(\widehat{P}_t) = \emptyset$ and moreover, from the known fact that $S_r(\widehat{P}_t) \cup S_p(\widehat{P}_t) = S_p(P_t)$, we conclude that $(e^{-q^\alpha t})_{q>0} \subseteq S_r(\widehat{P}_t)$. $\square$

Next, we will characterize a sequence of the so-called residual functions associated to $S_r(\widehat{P}_t)$, by means of (weak) Fourier kernels. To this end, we first recall from [87] that a linear operator $\widehat{\mathcal{H}}$ is called a weak Fourier kernel if there exists a linear space $\mathcal{D}(\widehat{\mathcal{H}})$ dense in $L^2(\mathbb{R}_+)$ and $\mathcal{M}_{\widehat{\mathcal{H}}} : \frac{1}{2} + i\mathbb{R} \to \mathbb{C}$ such that, for any $f \in \mathcal{D}(\widehat{\mathcal{H}})$,

$$b \mapsto \mathcal{M}_{\widehat{\mathcal{H}}f}\left(\frac{1}{2} + ib\right) = \mathcal{M}_{\widehat{\mathcal{H}}}\left(\frac{1}{2} + ib\right)\mathcal{M}_f\left(\frac{1}{2} - ib\right) \in L^2(\mathbb{R}). \tag{2.41}$$

Theorem 2.5.1. *Let us write, for $s \in \mathbb{C}_{(0,1)}$,*

$$\mathcal{M}_{\widehat{\mathcal{H}}_\alpha}(s) \quad = \quad \frac{\Gamma\left(\frac{1}{\alpha}\right)\Gamma(s)}{\Gamma\left(\frac{1-s}{\alpha}\right)\Gamma\left(1-\frac{1}{\alpha}+\frac{s}{\alpha}\right)}. \tag{2.42}$$

Then the following statements hold.

1. *$\widehat{\mathcal{H}}_\alpha$ is a weak Fourier kernel and $\mathcal{D} \subseteq \mathcal{D}(\widehat{\mathcal{H}}_\alpha)$, where the linear space $\mathcal{D}$ is defined by*

$$\mathcal{D} = \left\{ f \in \mathrm{L}^2(\mathbb{R}_+);\ \left|\mathcal{M}_f\left(\frac{1}{2}+ib\right)\right| \overset{\pm\infty}{=} \mathrm{O}\left(|b|^{-\frac{1}{2}-\epsilon} e^{\frac{-(2-\alpha)\pi}{2\alpha}|b|}\right) \text{ for some } \epsilon > 0 \right\}. \tag{2.43}$$

   Moreover, we have $\widehat{\mathcal{H}}_\alpha \mathcal{H}_\alpha f = f$ on $\mathrm{L}^2(\mathbb{R}_+)$ and $\mathcal{H}_\alpha \widehat{\mathcal{H}}_\alpha g = g$ on $\mathcal{D}(\widehat{\mathcal{H}}_\alpha)$. Consequently, $\widehat{\mathcal{H}}_\alpha$ is a self-adjoint operator on $\mathcal{D}(\widehat{\mathcal{H}}_\alpha)$.

2. *We have $\mathrm{Ran}(\Lambda_\alpha) \subseteq \mathcal{D}(\widehat{\mathcal{H}}_\alpha)$, and, on $\mathrm{L}^2(\mathbb{R}_+)$, $\widehat{\mathcal{H}}_\alpha \Lambda_\alpha = H_\alpha$ and $\mathcal{H}_\alpha = \Lambda_\alpha H_\alpha$.*

3. *For any $f \in \mathrm{L}^2(\mathbb{R}_+)$ and $t, q > 0$, we have $\widehat{\mathcal{H}}_\alpha P_t \Lambda_\alpha f(q) = e^{-q^\alpha t}\widehat{\mathcal{H}}_\alpha \Lambda_\alpha f(q)$. Moreover, for any $1 < \kappa < \frac{\alpha}{2-\alpha}$, $\mathcal{E}_\kappa = \mathrm{Span}(\mathbf{e}_{\kappa,\tau})_{\tau>0}$ is a dense subset of $\mathrm{L}^2(\mathbb{R}_+)$, and, for all $f \in \mathcal{E}_\kappa$, we have the integral representation, for almost every (a.e.) $q > 0$,*

$$\widehat{\mathcal{H}}_\alpha f(q) = \int_0^\infty f(x)\widehat{\mathcal{J}}_\alpha(qx)dx \tag{2.44}$$

   where, for $|\arg(z)| < \pi$, we set, with $\pi_\alpha = \frac{\pi}{\alpha}$,

$$\widehat{\mathcal{J}}_\alpha(z) \quad = \quad \frac{\Gamma\left(\frac{1}{\alpha}\right)}{\pi}\sin\left(\pi_\alpha - z\sin\left(\pi_\alpha\right)\right)e^{-z\cos(\pi_\alpha)}. \tag{2.45}$$

   We say that $\mathrm{d}_q\widehat{\mathcal{J}}_\alpha$ is a residual function for $\widehat{P}_t$ (or co-residual function for P_t) associated to residual spectrum value $e^{-q^\alpha t}$.

Proof. First, since from (2.26) and $0 < a < 1$ fixed, $\left|\mathcal{M}_{\widehat{\mathcal{H}}_\alpha}(a+ib)\right| \overset{\pm\infty}{=} \mathrm{O}\left(|b|^{a-\frac{1}{2}}e^{\frac{(2-\alpha)\pi}{2\alpha}|b|}\right)$, we deduce from (2.43) and the fact that, for all $b \in \mathbb{R}$, $\left|\mathcal{M}_f\left(\frac{1}{2}-ib\right)\right| = \left|\mathcal{M}_f\left(\frac{1}{2}+ib\right)\right|$, that

$$b \mapsto \left|\mathcal{M}_{\widehat{\mathcal{H}}_\alpha}\left(\frac{1}{2}+ib\right)\mathcal{M}_f\left(\frac{1}{2}-ib\right)\right| \overset{\pm\infty}{=} \mathrm{O}\left(|b|^{-\frac{1}{2}-\epsilon}\right) \in \mathrm{L}^2(\mathbb{R}).$$

Therefore $\mathcal{D} \subseteq \mathcal{D}(\widehat{\mathcal{H}}_\alpha)$. Next, observing that for any $1 < \kappa < \frac{\alpha}{2-\alpha}$, $\tau > 0$ and $s \in \mathbb{C}_{(0,\infty)}$,

$$\mathcal{M}_{\mathbf{e}_{\kappa,\tau}}(s) = \int_0^\infty x^{s-1}e^{-\tau x^\kappa}dx = \tau^{-\frac{s}{\kappa}}\kappa^{-1}\Gamma\left(\frac{s}{\kappa}\right), \tag{2.46}$$

we get $\left|\mathcal{M}_{\mathbf{e}_{\kappa,\tau}}\left(\frac{1}{2} - ib\right)\right| \stackrel{\pm\infty}{=} \mathrm{O}\left(|b|^{\frac{1}{2\kappa}-\frac{1}{2}}e^{-\frac{\pi}{2\kappa}|b|}\right)$, and thus, for any $1 < \kappa < \frac{\alpha}{2-\alpha}$, $(\mathbf{e}_{\kappa,\tau})_{\tau>0} \subseteq \mathcal{D}$. Moreover, since $(\mathbf{e}_{\kappa,\tau})_{\tau>0}$ is dense in $\mathrm{L}^2(\mathbb{R}_+)$, we obtain that $\mathcal{D}(\widehat{\mathcal{H}}_\alpha)$ is dense in $\mathrm{L}^2(\mathbb{R}_+)$ and therefore $\widehat{\mathcal{H}}_\alpha$ is a weak Fourier kernel. Next, using the definition of $\mathcal{H}_\alpha f$ in (2.36), we get, by performing a change of variable in (2.18), that for any $f \in \mathrm{L}^2(\mathbb{R}_+)$ and $s \in \frac{1}{2} + i\mathbb{R}$,

$$\mathcal{M}_{\mathcal{H}_\alpha f}(s) = \mathcal{M}_{\mathcal{J}_\alpha}(s)\mathcal{M}_f(1-s) = \frac{\Gamma\left(1-\frac{s}{\alpha}\right)\Gamma\left(\frac{s}{\alpha}\right)}{\Gamma\left(\frac{1}{\alpha}\right)\Gamma(1-s)}\mathcal{M}_f(1-s). \tag{2.47}$$

Thus, for such s, we have

$$\mathcal{M}_{\widehat{\mathcal{H}}_\alpha}(s)\mathcal{M}_{\mathcal{H}_\alpha f}(1-s) = \frac{\Gamma\left(\frac{1}{\alpha}\right)\Gamma(s)}{\Gamma\left(\frac{1-s}{\alpha}\right)\Gamma\left(1-\frac{1}{\alpha}+\frac{s}{\alpha}\right)}\mathcal{M}_{\mathcal{H}_\alpha f}(1-s) = \mathcal{M}_f(s).$$

Therefore, an application of the Parseval identity yields that $\mathcal{H}_\alpha f \in \mathcal{D}(\widehat{\mathcal{H}}_\alpha)$ and $\widehat{\mathcal{H}}_\alpha\mathcal{H}_\alpha f = f$ for all $f \in \mathrm{L}^2(\mathbb{R}_+)$. Similarly, one gets that $\mathcal{H}_\alpha\widehat{\mathcal{H}}_\alpha g = g$ for all $g \in \mathcal{D}(\widehat{\mathcal{H}}_\alpha)$. Next, from (2.36) one gets readily that $\mathcal{H}_\alpha$ is self-adjoint in $\mathrm{L}^2(\mathbb{R}_+)$, hence $\widehat{\mathcal{H}}_\alpha$ is also self-adjoint as the inverse operator of $\mathcal{H}_\alpha$, which concludes the proof of Theorem 2.5.1(1). Next, from (2.18), we have, for any $f \in \mathrm{L}^2(\mathbb{R}_+)$, $\mathcal{M}_{\Lambda_\alpha f}(s) = \mathcal{M}_f(s)\mathcal{M}_{\Lambda_\alpha}(1-s)$, therefore for at least $s \in \frac{1}{2} + i\mathbb{R}$,

$$\begin{aligned}\mathcal{M}_{\widehat{\mathcal{H}}_\alpha}(s)\mathcal{M}_{\Lambda_\alpha f}(1-s) &= \frac{\Gamma\left(\frac{1}{\alpha}\right)\Gamma(s)}{\Gamma\left(\frac{1-s}{\alpha}\right)\Gamma\left(1-\frac{1}{\alpha}+\frac{s}{\alpha}\right)}\mathcal{M}_{\Lambda_\alpha}(s)\mathcal{M}_f(1-s)\\ &= \frac{\Gamma(\frac{s}{\alpha})}{\Gamma(\frac{1-s}{\alpha})}\mathcal{M}_f(1-s) = \mathcal{M}_{H_\alpha f}(s)\end{aligned}$$

where the Mellin transform of $H_\alpha f$ is given in (2.66). Since from Proposition 2.7.1, $H_\alpha f \in \mathrm{L}^2(\mathbb{R}_+)$, we get, by the Parseval identity, that $\Lambda_\alpha f \in \mathcal{D}(\widehat{\mathcal{H}}_\alpha)$ and $\widehat{\mathcal{H}}_\alpha\Lambda_\alpha f = H_\alpha f$ for any $f \in \mathrm{L}^2(\mathbb{R}_+)$. Combine this relation with the self-inverse property of H_α from Proposition 2.7.1, we have, for any $f \in \mathrm{L}^2(\mathbb{R}_+)$, that

$\widehat{\mathcal{H}}_\alpha \Lambda_\alpha H_\alpha f = H_\alpha H_\alpha f = f$, which implies $\mathcal{H}_\alpha = \Lambda_\alpha H_\alpha$ and finishes the proof of Theorem 2.5.1(2). Next, combining the intertwining relation (2.21) and the spectral expansion (2.68) for Q_t, we get that, for any $f \in \mathrm{L}^2(\mathbb{R}_+), t > 0$,

$$P_t \Lambda_\alpha f = \Lambda_\alpha Q_t f = \Lambda_\alpha H_\alpha \mathbf{e}_{\alpha,t} H_\alpha f = \mathcal{H}_\alpha \mathbf{e}_{\alpha,t} H_\alpha f. \tag{2.48}$$

Hence, by observing that $P_t \Lambda_\alpha f \in \mathrm{Ran}(\mathcal{H}_\alpha)$ and by means of Theorem 2.5.1(1), we get

$$\widehat{\mathcal{H}}_\alpha P_t \Lambda_\alpha f = \widehat{\mathcal{H}}_\alpha \mathcal{H}_\alpha \mathbf{e}_{\alpha,t} H_\alpha f = \mathbf{e}_{\alpha,t} H_\alpha f = \mathbf{e}_{\alpha,t} \widehat{\mathcal{H}}_\alpha \Lambda_\alpha f.$$

Finally, since, from above, we have, for any $1 < \kappa < \frac{\alpha}{2-\alpha}$, $(\mathbf{e}_{\kappa,\tau})_{\tau>0} \subseteq \mathcal{D}$, we get that $\widehat{\mathcal{H}}_\alpha \mathbf{e}_{\kappa,\tau} \in \mathrm{L}^2(\mathbb{R}_+)$ and by combining (2.42) with (2.46), that, for at least $s \in \frac{1}{2} + i\mathbb{R}$,

$$\mathcal{M}_{\widehat{\mathcal{H}}_\alpha \mathbf{e}_{\kappa,\tau}}(s) = \frac{\tau^{\frac{s-1}{\kappa}} \Gamma(\frac{1}{\alpha}) \Gamma(s) \Gamma(\frac{1-s}{\kappa})}{\kappa \Gamma(1 - \frac{1}{\alpha} + \frac{s}{\alpha}) \Gamma(\frac{1-s}{\alpha})}.$$

By following a line of reasoning similar to the one used in the proof of Lemma 2.3.1, we obtain

$$\widehat{\mathcal{H}}_\alpha \mathbf{e}_{\kappa,\tau}(q) = \frac{\Gamma(\frac{1}{\alpha})}{\kappa} \sum_{n=0}^{\infty} \frac{(-1)^n \tau^{-\frac{n+1}{\kappa}} \Gamma(\frac{n+1}{\kappa})}{n! \Gamma(1 - \frac{1}{\alpha} - \frac{n}{\alpha}) \Gamma(\frac{n+1}{\alpha})} q^n = \frac{\Gamma\left(\frac{1}{\alpha}\right)}{\kappa \pi} \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma\left(\frac{n+1}{\kappa}\right) \sin\left((n+1)\pi_\alpha\right)}{\tau^{\frac{n+1}{\kappa}} n!} q^n,$$

which defines an entire function since, by the Stirling approximation (2.25), $\frac{\Gamma(\frac{n+1}{\kappa})}{n\Gamma(\frac{n}{\kappa})} \stackrel{\infty}{=} \mathrm{O}\left(n^{\frac{1}{\kappa}-1}\right)$ and $\kappa > 1$. On the other hand, since for any $x > 0$,

$$\frac{\pi}{\Gamma\left(\frac{1}{\alpha}\right)} \widehat{\mathcal{J}}_\alpha(x) = \cos\left(\pi_\alpha\right) \Im\left(e^{-xe^{i\pi_\alpha}}\right) + \sin\left(\pi_\alpha\right) \Re\left(e^{-xe^{i\pi_\alpha}}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \sin\left((n+1)\pi_\alpha\right) x^n,$$

a standard application of Fubini's theorem, see again [113, Section 1.77], yields

$$\begin{aligned}
\int_0^\infty \mathbf{e}_{\kappa,\tau}(x) \widehat{\mathcal{J}}_\alpha(qx) dx &= \frac{\Gamma\left(\frac{1}{\alpha}\right)}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \sin\left((n+1)\pi_\alpha\right) q^n \int_0^\infty e^{-\tau x^\kappa} x^n dx \\
&= \frac{\Gamma\left(\frac{1}{\alpha}\right)}{\kappa \pi} \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma\left(\frac{n+1}{\kappa}\right) \sin\left((n+1)\pi_\alpha\right)}{\tau^{\frac{n+1}{\kappa}} n!} q^n = \widehat{\mathcal{H}}_\alpha \mathbf{e}_{\kappa,\tau}(q),
\end{aligned}$$

from which we conclude that $\widehat{\mathcal{H}}_\alpha f(q) = \int_0^\infty f(x) \widehat{\mathcal{J}}_\alpha(qx) dx$ for all $f \in \mathcal{E}_\kappa$. This completes the proof. □

2.6 Spectral representation, heat kernel and smoothness properties

We have now all the ingredients for stating and proving the spectral representation of the semigroups P and $\widehat{P}$ along with the representation of the heat kernel.

2.6.1 Spectral expansions of P and $\widehat{P}$ in Hilbert spaces and the heat kernel

Theorem 2.6.1. *1. For any $g \in \mathrm{L}^2(\mathbb{R}_+)$ and $t > 0$, we have in $\mathrm{L}^2(\mathbb{R}_+)$*

$$\widehat{P}_t g = \widehat{\mathcal{H}}_\alpha \mathbf{e}_{\alpha,t} \mathcal{H}_\alpha g. \tag{2.49}$$

2. The heat kernel of P admits the representation

$$P_t(x,y) = \int_0^\infty e^{-q^\alpha t} \mathcal{J}_\alpha(qx) \widehat{\mathcal{J}}_\alpha(qy) dq, \tag{2.50}$$

where the integral is locally uniformly convergent in $(t,x,y) \in \mathbb{R}_+^3$.

3. For any $t > T_\alpha$, we have in $\mathrm{L}^2(\mathbb{R}_+)$,

$$P_t f = \mathcal{H}_\alpha \mathbf{e}_{\alpha,t} \widehat{\mathcal{H}}_\alpha f \tag{2.51}$$

where

(a) if $f \in \mathcal{D}(\widehat{\mathcal{H}}_\alpha)$ then $T_\alpha = 0$,

(b) otherwise if $f \in \mathrm{L}^2\left(\overline{\mathbf{e}}_{\kappa,\eta}\right)$ for some $\kappa \geq \frac{\alpha}{\alpha-1}$ and $\eta > 0$, where we set $\overline{\mathbf{e}}_{\kappa,\eta}(x) = e^{\eta x^\kappa}, x > 0$, then $T_\alpha = \frac{\eta}{\alpha-1}\left(2\frac{\alpha-1}{\alpha\eta}\cos((\alpha+1)\pi_\alpha)\right)^\alpha \mathbb{I}_{\{\kappa(\alpha-1)=\alpha\}}$ and $\widehat{\mathcal{H}}_\alpha f(q) = \int_0^\infty f(y)\widehat{\mathcal{J}}_\alpha(qy)dy$.

Remark 2.6.1. *We mention that* $\mathcal{D}(\widehat{\mathcal{H}}_\alpha)\backslash \mathrm{L}^2\left(\overline{\mathbf{e}}_{\kappa,\eta}\right) \neq \emptyset$ *meaning that the two conditions* (3a) *and* (3b) *are applicable under different situations. For instance, for* $0 < \beta < \min\left(\frac{\alpha}{2-\alpha}, \frac{\alpha}{\alpha-1}\right)$, $\mathbf{e}_\beta \in \mathrm{Ran}(\Lambda_\alpha)\backslash \mathrm{L}^2\left(\overline{\mathbf{e}}_{\kappa,\eta}\right)$, *as one can show that* $\Lambda_\alpha B_\beta = \mathbf{e}_\beta$ *with* $x \mapsto B_\beta(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\mathcal{M}_{\Lambda_\alpha}(\beta n+1)n!} x^{\beta n} \in \mathrm{L}^2(\mathbb{R}_+)$.

Proof. First, since for any $f \in \mathrm{L}^2(\mathbb{R}_+)$, $\widehat{P}_t f \in \mathrm{L}^2(\mathbb{R}_+)$, we get, for all $q > 0$,

$$\mathcal{H}_\alpha \widehat{P}_t f(q) = \left\langle \widehat{P}_t f, \mathrm{d}_q \mathcal{J}_\alpha \right\rangle = \left\langle f, P_t \mathrm{d}_q \mathcal{J}_\alpha \right\rangle = e^{-q^\alpha t} \left\langle f, \mathrm{d}_q \mathcal{J}_\alpha \right\rangle = e^{-q^\alpha t} \mathcal{H}_\alpha f(q)$$

where we used Theorem 2.4.1(1) for $\mathrm{d}_q \mathcal{J}_\alpha \in \mathrm{L}^2(\mathbb{R}_+)$ and for the third identity. Therefore, we can apply Theorem 2.5.1(1) to get that for any $g \in \mathrm{L}^2(\mathbb{R}_+)$,

$$\widehat{P}_t g = \widehat{\mathcal{H}}_\alpha \mathcal{H}_\alpha \widehat{P}_t g = \widehat{\mathcal{H}}_\alpha \mathbf{e}_{\alpha,t} \mathcal{H}_\alpha g,$$

which proves (2.49). On the other hand, for any $f \in \mathcal{D}(\widehat{\mathcal{H}}_\alpha)$, $g \in \mathrm{L}^2(\mathbb{R}_+)$, we have, using the self-adjoint property of $\widehat{\mathcal{H}}_\alpha$ and $\mathcal{H}_\alpha$, see Theorem 2.5.1(1),

$$\langle P_t f, g \rangle = \left\langle f, \widehat{P}_t g \right\rangle = \left\langle f, \widehat{\mathcal{H}}_\alpha \mathbf{e}_{\alpha,t} \mathcal{H}_\alpha g \right\rangle = \left\langle \mathcal{H}_\alpha \mathbf{e}_{\alpha,t} \widehat{\mathcal{H}}_\alpha f, g \right\rangle,$$

which proves (2.51) for $f \in \mathcal{D}(\widehat{\mathcal{H}}_\alpha)$ and $T_\alpha = 0$, that is, the claim (3a). Next, let us consider the density function $\lambda_{\mathbf{X}_\alpha} \in \mathrm{L}^2(\mathbb{R}_+)$ of the random variable $\mathbf{X}_\alpha$, which we recall was studied in Lemma 2.3.2. Then using (2.47) again, it is easy to deduce that $\mathcal{M}_{\mathcal{H}_\alpha \lambda_{\mathbf{X}_\alpha}}(s) = \frac{\Gamma(\frac{s}{\alpha})}{\Gamma(\frac{1}{\alpha})}$, which coincides with the Mellin transform of G_α (see (2.63)). Hence we have, for all $q > 0$, that $\mathcal{H}_\alpha \lambda_{\mathbf{X}_\alpha}(q) = \lambda_{G_\alpha}(q) = \frac{e^{-q^\alpha}}{\Gamma(1+\frac{1}{\alpha})}$. Therefore, we see that for any $\tau > 0$, $q \mapsto \mathbf{e}_{\alpha,t} \mathcal{H}_\alpha \mathrm{d}_\tau \lambda_{\mathbf{X}_\alpha}(q) = \frac{e^{-(\tau^{-\alpha}+t)q^\alpha}}{\tau \Gamma(1+\frac{1}{\alpha})} \in \mathcal{E}_\alpha$. Hence, using Theorem 2.5.1(3), we can write

$$\widehat{P}_t \mathrm{d}_\tau \lambda_{\mathbf{X}_\alpha}(y) = \widehat{\mathcal{H}}_\alpha \mathbf{e}_{\alpha,t} \mathcal{H}_\alpha \mathrm{d}_\tau \lambda_{\mathbf{X}_\alpha}(y) = \int_0^\infty e^{-q^\alpha t} \widehat{\mathcal{J}}_\alpha(qy) \int_0^\infty \lambda_{\mathbf{X}_\alpha}(\tau x) \mathcal{J}_\alpha(qx) dx dq. \quad (2.52)$$

Next, from (2.37) we deduce that $|\mathcal{J}_\alpha(x)| \stackrel{0}{=} \mathrm{O}(1)$ and $|\mathcal{J}_\alpha(x)| \stackrel{\infty}{=} \mathrm{O}(x^{-\alpha})$ and thus, since $\lambda_{\mathbf{X}_\alpha}$ is a probability density function, $\int_0^\infty \lambda_{\mathbf{X}_\alpha}(\tau x) |\mathcal{J}_\alpha(qx)| dx \leq C(1 + q^{-\alpha})$ for some $C = C(\tau) > 0$. On the other hand, from (2.45), we get that there exists $\hat{C} > 0$ such that for all $y > 0$, $|\widehat{\mathcal{J}}_\alpha(y)| \leq \hat{C}e^y$, which justifies an application of Fubini theorem to obtain

$$\int_0^\infty e^{-q^\alpha t}\widehat{\mathcal{J}}_\alpha(qy)\int_0^\infty \lambda_{\mathbf{X}_\alpha}(\tau x)\mathcal{J}_\alpha(qx)dxdq = \int_0^\infty \lambda_{\mathbf{X}_\alpha}(\tau x)\int_0^\infty e^{-q^\alpha t}\mathcal{J}_\alpha(qx)\widehat{\mathcal{J}}_\alpha(qy)dqdx. \tag{2.53}$$

Now let us define the Mellin convolution operator $\mathcal{X}$ by $\mathcal{X}f(\tau) = \int_0^\infty f(y)\lambda_{\mathbf{X}_\alpha}(\tau y)dy$ and, since $\lambda_{\mathbf{X}_\alpha} \in \mathrm{L}^2(\mathbb{R}_+)$, $\mathcal{X} \in \mathbf{B}\mathrm{L}^2(\mathbb{R}_+)$ and by performing a change of variable in (2.18) we get, from (2.33), $\mathcal{M}_{\mathcal{X}}(s) = \mathcal{M}_{\mathbf{X}_\alpha}(s) = \frac{\Gamma(s)}{\Gamma(\frac{s}{\alpha}+1-\frac{1}{\alpha})}$ which is clearly zero-free on $\Re(s) = 1$ entailing that $\mathcal{X}$ is one-to-one in $\mathrm{L}^2(\mathbb{R}_+)$. Moreover, by means of the same upper bounds used above, we deduce that for any y fixed,

$$x \mapsto \int_0^\infty e^{-q^\alpha t}\mathcal{J}_\alpha(qx)\widehat{\mathcal{J}}_\alpha(qy)dq \in \mathrm{L}^2(\mathbb{R}_+)$$

and thus the right-hand side of (2.53) is in $\mathrm{L}^2(\mathbb{R}_+)$ and hence from (2.52), we get that, for any $\tau > 0$,

$$\widehat{P}_t \mathrm{d}_\tau \lambda_{\mathbf{X}_\alpha}(y) = \int_0^\infty \lambda_{\mathbf{X}_\alpha}(\tau x)\int_0^\infty e^{-q^\alpha t}\mathcal{J}_\alpha(qx)\widehat{\mathcal{J}}_\alpha(qy)dqdx.$$

The one-to-one property of $\mathcal{X}$ implies that the transition kernel of $\widehat{P}_t$, denoted by $\widehat{P}_t(y,x)$, can be represented, for a.e. $y > 0$, as $\widehat{P}_t(y,x) = \int_0^\infty e^{-q^\alpha t}\mathcal{J}_\alpha(qx)\widehat{\mathcal{J}}_\alpha(qy)dq$. Since the last integral is also locally uniformly convergent for any $(t,x,y) \in \mathbb{R}_+^3$, and $\widehat{\mathcal{J}}_\alpha$ is continuous, the identity holds everywhere. This last fact combined with the duality stated in Proposition 2.2.1(3) yield the expression (2.50) by recalling, from Proposition 2.2.1(3), that since the Lebesgue measure serves as reference measure we get that $P_t(x,y) = \widehat{P}_t(y,x)$, $t,x,y > 0$. While (3a) has been proved above, we now proceed to the justification of (3b). First, by the Cauchy-Schwarz inequality, observe that for any $f \in \mathrm{L}^2\left(\overline{\mathbf{e}}_{\kappa,\eta}\right)$, writing $\widehat{J_{\overline{\mathbf{e}}}}(qy) = \frac{\widehat{\mathcal{J}}_\alpha(qy)}{\overline{\mathbf{e}}_{\kappa,\eta}(y)}$, we have

$$\int_0^\infty f(y)\widehat{\mathcal{J}}_\alpha(qy)dy = \int_0^\infty f(y)\widehat{J_{\overline{\mathbf{e}}}}(qy)\overline{\mathbf{e}}_{\kappa,\eta}(y)dy \leq \|f\|_{\overline{\mathbf{e}}_{\kappa,\eta}}\left\|\mathrm{d}_q\widehat{J_{\overline{\mathbf{e}}}}\right\|_{\overline{\mathbf{e}}_{\kappa,\eta}}.$$

Moreover, since for all $y > 0$, $\left|\widehat{\mathcal{J}}_\alpha(y)\right| \leq Ce^{-y\cos(\pi_\alpha)}$, $C > 0$, we have by an application of the Laplace method, see e.g. [80, Ex.7.3 p.84], that for large q,

$$\left\|\mathrm{d}_q\widehat{\mathcal{J}}_{\overline{\mathbf{e}}}\right\|^2_{\overline{\mathbf{e}}_{\kappa,\eta}} \leq C^2\int_0^\infty e^{-\eta y^\kappa}e^{-2q\cos(\pi_\alpha)y}dy \overset{\infty}{=} \mathrm{O}\left(q^a e^{c_\kappa q^{\frac{\kappa}{\kappa-1}}}\right),$$

where $a > 0$ and we set $c_\kappa = (\kappa-1)\eta^{\frac{1}{1-\kappa}}\left(\frac{2\cos((\alpha+1)\pi_\alpha)}{\kappa}\right)^{\frac{\kappa}{\kappa-1}} > 0$ since $\kappa > \alpha > 1$. Note that $c_{\frac{\alpha}{\alpha-1}} = T_\alpha$ and for any $t > T_\alpha$, since $\alpha \geq \frac{\kappa}{\kappa-1}$, $q \mapsto F_\kappa(q) = \mathbf{e}_{\alpha,t}(q)\left(C + q^a e^{c_\kappa q^{\frac{\kappa}{\kappa-1}}}\right)$ is integrable on $\mathbb{R}_+$. This justifies an application of Fubini Theorem which gives that, for such f, t and $x > 0$,

$$P_t f(x) = \int_0^\infty f(y)\int_0^\infty e^{-q^\alpha y}\mathcal{J}_\alpha(qx)\widehat{\mathcal{J}}_\alpha(qy)dqdy. \tag{2.54}$$

Finally, as $F_\kappa \in \mathrm{L}^2(\mathbb{R}_+)$ and from Theorem 3.2.1, the sequence $(\mathrm{d}_q\mathcal{J}_\alpha)_{q>0}$ is an upper frame, we obtain that in fact $P_t f \in \mathrm{L}^2(\mathbb{R}_+)$ and, in $\mathrm{L}^2(\mathbb{R}_+)$, $P_t f = \mathcal{H}_\alpha\mathbf{e}_{\alpha,t}\widehat{\mathcal{H}}_\alpha f$ with $\widehat{\mathcal{H}}_\alpha f(q) = \int_0^\infty f(y)\widehat{\mathcal{J}}_\alpha(qy)dy$. This completes the proof. □

2.6.2 Regularity properties

Finally, we extract from the spectral decomposition stated in Theorem 2.6.1 the following regularity properties as well as an alternative representation of the heat kernel.

Theorem 2.6.2. *1. For any $f \in \mathrm{L}^2\left(\overline{\mathbf{e}}_{\kappa,\eta}\right) \cup \mathcal{D}(\widehat{\mathcal{H}}_\alpha)$, $(t,x) \mapsto P_t f(x) \in \mathrm{C}^\infty((T_\alpha,\infty)\times\mathbb{R}_+)$ and T_α was defined in Theorem 3.2.1.*

2. We have $(t,x,y) \mapsto P_t(x,y) \in \mathrm{C}^\infty(\mathbb{R}^3_+)$ and, for any non-negative integers $k, \mathfrak{p}, \mathfrak{q}$,

$$\frac{d^k}{dt^k}P_t^{(\mathfrak{p},\mathfrak{q})}(x,y) = (-1)^k\int_0^\infty q^{\alpha k}e^{-q^\alpha t}\left(\mathrm{d}_q\mathcal{J}_\alpha\right)^{(\mathfrak{p})}(x)\left(\mathrm{d}_q\widehat{\mathcal{J}}_\alpha\right)^{(\mathfrak{q})}(y)dq \tag{2.55}$$

where the integral is locally uniformly convergent in $(t,x,y) \in \mathbb{R}^3_+$.

3. *Moreover, the heat kernel can be written in a series form as*

$$P_t(x,y) = \sum_{n=0}^{\infty}(1+t)^{-n-\frac{1}{\alpha}}\mathcal{P}_n(x^\alpha)\mathcal{V}_n\left(y(1+t)^{-\frac{1}{\alpha}}\right), \tag{2.56}$$

where $\mathcal{P}_n(x) = \frac{1}{\Gamma(1+\frac{1}{\alpha})}\sum_{k=0}^{n}(-1)^k\frac{\binom{n}{k}k!}{\Gamma(\alpha k+1)}x^k$ *and* $\mathcal{V}_n(y) = \frac{1}{n!}\int_0^\infty q^{\alpha n}e^{-q^\alpha}\widehat{\mathcal{J}}_\alpha(qy)dq$ *and the series is locally uniformly convergent in* $(t,x,y) \in \mathbb{R}_+^3$.

Proof. We actually prove only the item (2) as the first item follows by developing similar arguments. First, from Theorem 2.4.1 and Theorem 2.5.1, we have that $\mathcal{J}_\alpha, \widehat{\mathcal{J}}_\alpha \in C^\infty(\mathbb{R}_+)$ and for any $x,y>0$ fixed and non-negative integers $k, \mathfrak{p}, \mathfrak{q}$,

$$\left|\frac{d^k}{dt^k}e^{-q^\alpha t}(\mathrm{d}_q\mathcal{J}_\alpha)^{(\mathfrak{p})}(x)(\mathrm{d}_q\widehat{\mathcal{J}}_\alpha)^{(\mathfrak{q})}(y)\right| \stackrel{\infty}{=} \mathrm{O}\left(q^{\alpha(k-1)+\mathfrak{q}}e^{-q^\alpha t+qy}\right)$$

and

$$\left|\frac{d^k}{dt^k}e^{-q^\alpha t}(\mathrm{d}_q\mathcal{J}_\alpha)^{(\mathfrak{p})}(x)(\mathrm{d}_q\widehat{\mathcal{J}}_\alpha)^{(\mathfrak{q})}(y)\right| \stackrel{0}{=} \mathrm{O}\left(q^{\alpha k+\mathfrak{p}+\mathfrak{q}}\right).$$

Hence (2.50) yields

$$\frac{d^k}{dt^k}P_t^{(\mathfrak{p},\mathfrak{q})}(x,y) = (-1)^k\int_0^\infty q^{\alpha k}e^{-q^\alpha t}(\mathrm{d}_q\mathcal{J}_\alpha)^{(\mathfrak{p})}(x)(\mathrm{d}_q\widehat{\mathcal{J}}_\alpha)^{(\mathfrak{q})}(y)dq$$

where the integral is locally uniformly convergent in $(t,x,y) \in \mathbb{R}_+^3$. Hence $(t,x,y) \mapsto P_t(x,y) \in C^\infty(\mathbb{R}_+^3)$. To prove (2.56), we first observe, from [37, Proposition 2.1(ii)], that for any $x, q \in \mathbb{R}_+$,

$$e^{q^\alpha}\mathcal{J}_\alpha(qx) = \sum_{n=0}^{\infty}\mathcal{P}_n(x^\alpha)\frac{q^{\alpha n}}{n!}, \tag{2.57}$$

which by substitution in (2.50) gives, assuming, for a moment, that one may interchange the sum and integral,

$$P_t(x,y) \;=\; \int_0^\infty e^{-q^\alpha(t+1)}\widehat{\mathcal{J}}_\alpha(qy)\sum_{n=0}^{\infty}\frac{\mathcal{P}_n(x^\alpha)}{n!}q^{\alpha n}dq = \sum_{n=0}^{\infty}(1+t)^{-n-\frac{1}{\alpha}}\mathcal{P}_n(x^\alpha)\mathcal{V}_n\left(y(1+t)^{-\frac{1}{\alpha}}\right).$$

In order to justify the interchange we provide some uniform bounds for large n of $\mathcal{P}_n$ and $\mathcal{V}_n$. First, since $z \mapsto \mathcal{J}_\alpha(z^{\frac{1}{\alpha}})$ is an entire function of order $\overline{\lim}_{n\to\infty}\frac{n\ln n}{\Gamma(\alpha n+1)} =$

$\frac{1}{\alpha}$ and type 1, by following a line of reasoning similar to the proof of [91, Theorem 8.4(5)], we obtain the following sequence of inequalities, valid for all $x > 0$ and n large,

$$\begin{aligned} |\mathcal{P}_n(x)| \leq \mathcal{P}_n(-x) &= \frac{n!}{2\pi i} x^n \oint_{nx} e^{\frac{z}{x}} \mathcal{J}_\alpha\left(-z^{\frac{1}{\alpha}}\right) \frac{dz}{z^{n+1}} \\ &\leq e^{n^{\frac{1}{\alpha}} x^{\frac{1}{\alpha}}} \frac{n! e^{-n \ln n}}{2\pi} \int_0^{2\pi} e^{n \cos\theta} d\theta = \mathrm{O}\left(n^{\frac{1}{2}} e^{(xn)^{\frac{1}{\alpha}}}\right) \end{aligned}$$

where the contour is a circle centered at 0 with radius $nx > 0$ and for the last inequality we used the bound $n! \leq e^{1-n} n^{n-\frac{1}{2}}$. Hence, we have, for all fixed $x > 0$ and n large,

$$|\mathcal{P}_n(x^\alpha)| = \mathrm{O}\left(n^{\frac{1}{2}} e^{xn^{\frac{1}{\alpha}}}\right). \tag{2.58}$$

Next, since for any $q > 0$, $|\widehat{\mathcal{J}}_\alpha(q)| \leq \widehat{\mathcal{J}}_\alpha(-q) \leq Ce^q$, for some constant $C = C(\alpha) > 0$, we get, for all $y > 0$ and $n \in \mathbb{N}$,

$$|\mathcal{V}_n(y)| \leq \frac{C}{n!} \int_0^\infty e^{-q^\alpha} q^{\alpha n} \sum_{k=0}^\infty \frac{(yq)^k}{k!} dq = C \sum_{k=0}^\infty \frac{\Gamma(n+1+\frac{k}{\alpha})}{n! k!} y^k$$

where for the equality we use the integral representation of the gamma function. Now, by performing the same computations that in the proof of [88, Proposition 2.2], we get, for all $y > 0$ and n large,

$$|\mathcal{V}_n(y)| = \mathrm{O}\left(n e^{\bar{c}_\alpha y n^{\frac{1}{\alpha}}}\right) \tag{2.59}$$

where $\bar{c}_\alpha^\alpha = \frac{\alpha}{\alpha-1}$. Hence combining the bounds (2.58) and (2.59), we obtain, for any fixed $x, y, t > 0$ and large n,

$$(1+t)^{-n-\frac{1}{\alpha}} \left|\mathcal{P}_n(x^\alpha) \mathcal{V}_n\left(y(1+t)^{-\frac{1}{\alpha}}\right)\right| = \mathrm{O}\left(n^{\frac{3}{2}} e^{(\bar{c}_\alpha y(1+t)^{-\frac{1}{\alpha}} + x) n^{\frac{1}{\alpha}} - \ln(1+t)n}\right), \tag{2.60}$$

which justifies the interchange and completes the proof. $\square$

2.7 The α-Bessel semigroup and the operator H_α

We say $Q = (Q_t)_{t\geq 0}$ is an α-Bessel semigroup with index $1 < \alpha < 2$ if it is a Feller semigroup whose infinitesimal generator is given by

$$\mathbf{L}f(x) = \frac{2}{\alpha^2}x^{2-\alpha}f^{(2)}(x) + \frac{2}{\alpha}\left(\frac{2}{\alpha}-1\right)x^{1-\alpha}f^{(1)}(x),\ x > 0, \tag{2.61}$$

where $f \in \mathcal{D}_{\mathbf{L}} = \{f \in \mathrm{C}_0(\mathbb{R}_+);\ \mathbf{L}f \in \mathrm{C}_0(\mathbb{R}_+), f^+(0) = 0\}$, the domain of $\mathbf{L}$, with $f^+(x) = \lim_{h\downarrow 0}\frac{f(x+h)-f(x)}{s(x+h)-s(x)}$ is the right-derivative of f with respect to the scale function $s(x) = \frac{x^{\alpha-1}}{\alpha-1}$. We point out that

$$Q_t f(x) = K_t \mathrm{p}_{\frac{1}{\alpha}} f(x^\alpha),\ x > 0, \tag{2.62}$$

where $K = (K_t)_{t\geq 0}$ is the semigroup of a squared Bessel process of dimension $\frac{2}{\alpha}$, or equivalently of order $\frac{1}{\alpha} - 1$ and $\mathrm{p}_{\frac{1}{\alpha}} f(x) = f(x^{\frac{1}{\alpha}})$. We refer in this part to [23, Appendix 1] for concise information on squared Bessel processes that can be easily transferred to Q by means of the identity (2.62). Furthermore, writing $(\vartheta_t)_{t>0}$ for the entrance law of Q, we have $\vartheta_t f = \int_0^\infty f(ty)\lambda_{G_\alpha}(y)dy$ where $\lambda_{G_\alpha}(y) = \frac{e^{-y^\alpha}}{\Gamma(1+\frac{1}{\alpha})}, y > 0$, is the density of the variable G_α. Note that G_α^α is simply a gamma variable of parameter $\frac{1}{\alpha}$, the law of this latter being the entrance law at time 1 of K. The Mellin transform of G_α is given by

$$\mathcal{M}_{G_\alpha}(s) = \frac{\Gamma(\frac{s}{\alpha})}{\Gamma(\frac{1}{\alpha})}, \quad \Re(s) > 0. \tag{2.63}$$

Next, defining the function J_α, for $z \in \mathbb{C}_{(-\infty,0)^c}$, by

$$J_\alpha(z) = \alpha\sum_{n=0}^{\infty}\frac{(e^{i\pi}z^\alpha)^n}{n!\Gamma(n+\frac{1}{\alpha})}, \tag{2.64}$$

we can deduce from [23, Appendix 1] that for any $q, t, x \geq 0$,

$$Q_t \mathrm{d}_q J_\alpha(x) = e^{-q^\alpha t}\mathrm{d}_q J_\alpha(x).$$

Next, we introduce the linear operator defined, for a smooth function f on $q > 0$, by

$$H_\alpha f(q) = \int_0^\infty J_\alpha(qx) f(x) dx. \tag{2.65}$$

Then, H_α has the following properties reminiscent of the classical Hankel transform.

Proposition 2.7.1. *H_α is a unitary and self-inverse operator on $\mathrm{L}^2(\mathbb{R}_+)$, i.e. $\|H_\alpha f\| = \|f\|$ and $H_\alpha H_\alpha f = f$ for all $f \in \mathrm{L}^2(\mathbb{R}_+)$. Moreover, for any $f \in \mathrm{L}^2(\mathbb{R}_+)$, the Mellin transform of $H_\alpha f$ is given by*

$$\mathcal{M}_{H_\alpha f}(s) = \mathcal{M}_{J_\alpha}(s)\mathcal{M}_f(1-s) = \frac{\Gamma(\frac{s}{\alpha})}{\Gamma(\frac{1-s}{\alpha})}\mathcal{M}_f(1-s), \quad s \in \mathbb{C}_{(0,1)}. \tag{2.66}$$

Proof. First, note that $J_\alpha(x) = \alpha x^{\frac{\alpha-1}{2}} \mathtt{J}_{\frac{1}{\alpha}-1}\left(2x^{\frac{\alpha}{2}}\right)$ where $\mathtt{J}_{\frac{1}{\alpha}-1}$ denotes the standard Bessel function of the first kind of order $\frac{1}{\alpha} - 1$, see e.g. [70, Section 5.3]. Then recall that the standard Hankel transform is defined, for any $g \in \mathrm{L}^2(m)$ where $m(dx) = xdx$, as

$$\mathtt{H}_\alpha g(r) = \int_0^\infty \mathtt{J}_{\frac{1}{\alpha}-1}(rx) g(x) x dx, \quad r > 0.$$

Then by [95, Chapter 9], $\mathtt{H}_\alpha$ is unitary and self-inverse on $\mathrm{L}^2(m)$, i.e. for any $g \in \mathrm{L}^2(m)$, we have $\|\mathtt{H}_\alpha g\|_m = \|g\|_m$ and $\mathtt{H}_\alpha\mathtt{H}_\alpha g = g$. Now for any $f \in \mathrm{L}^2(\mathbb{R}_+)$, we set $g(x) = x^{\frac{1}{\alpha}-1} f\left((\frac{x}{2})^{\frac{2}{\alpha}}\right)$. Then it can be easily checked, through a standard change of variable, that $g \in \mathrm{L}^2(m)$ and $\|g\|_m^2 = \alpha 2^{\frac{2}{\alpha}-1}\|f\|^2$. Therefore, by applying a change of variable, one gets

$$\begin{aligned}
\|H_\alpha f\|^2 &= \int_0^\infty \left|\int_0^\infty f(x) J_\alpha(qx) dx\right|^2 dq = \frac{1}{2^{\frac{2}{\alpha}}} \int_0^\infty \left|\int_0^\infty \mathtt{J}_{\frac{1}{\alpha}-1}(q^{\frac{\alpha}{2}} y) g(y) y dy\right|^2 \frac{dq}{q^{1-\alpha}} \\
&= \frac{1}{2^{\frac{2}{\alpha}}} \int_0^\infty q^{\alpha-1} \left|\mathtt{H}_\alpha g(q^{\frac{\alpha}{2}})\right|^2 dq = \frac{1}{\alpha 2^{\frac{2}{\alpha}-1}} \|\mathtt{H}_\alpha g\|_m^2 = \frac{1}{\alpha 2^{\frac{2}{\alpha}-1}} \|g\|_m^2 = \|f\|^2.
\end{aligned}$$

This proves that H_α is a unitary operator. Next, for any $f \in \mathrm{L}^2(\mathbb{R}_+)$, again by change of variable, we have

$$H_\alpha H_\alpha f(y) = \int_0^\infty J_\alpha(qy) \int_0^\infty f(x) J_\alpha(qx) dx dq = \frac{y^{\frac{\alpha-1}{2}}}{2^{\frac{1}{\alpha}-1}} \mathtt{H}_\alpha\mathtt{H}_\alpha g(2y^{\frac{\alpha}{2}}) = \frac{y^{\frac{\alpha-1}{2}}}{2^{\frac{1}{\alpha}-1}} g(2y^{\frac{\alpha}{2}}) = f(y),$$

which proves that H_α is self-inverse. Next, using again a change of variable in (2.18), we have $\mathcal{M}_{H_\alpha f}(s) = \mathcal{M}_{J_\alpha}(s)\mathcal{M}_f(1-s)$, where for $0 < \Re(s) < 1$, $\mathcal{M}_{J_\alpha}(s) = \frac{\Gamma(\frac{s}{\alpha})}{\Gamma(\frac{1-s}{\alpha})}$ can be proved by the Mellin-Barnes integral representation of Bessel functions, see e.g. [83, Section 3.4.3], which gives that

$$\frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} z^{-s}\frac{\Gamma(\frac{s}{\alpha})}{\Gamma(\frac{1-s}{\alpha})}ds = \alpha\sum_{n=0}^{\infty}\frac{(e^{i\pi}z^\alpha)^n}{n!\Gamma(n+\frac{1}{\alpha})} = J_\alpha(z).$$

This concludes the proof of the Proposition. □

Next, by referring to [23, Chapter II], we see that the speed measure of Q is (up to a multiplicative positive constant) the Lebesgue measure, hence Q extends uniquely to a self-adjoint contractive C_0-semigroup on $\mathrm{L}^2(\mathbb{R}_+)$, also denoted by Q when there is no confusion (otherwise, we may denote Q^F for the Feller semigroup). The infinitesimal generator $\mathbf{L}$ of this $\mathrm{L}^2(\mathbb{R}_+)$-extension is an unbounded self-adjoint operator on $\mathrm{L}^2(\mathbb{R}_+)$, and, by [73, Remark 3.1], its $\mathrm{L}^2(\mathbb{R}_+)$-domain, denoted by $\mathcal{D}_{\mathbf{L}}(\mathrm{L}^2(\mathbb{R}_+))$, is given by

$$\mathcal{D}_{\mathbf{L}}(\mathrm{L}^2(\mathbb{R}_+)) = \{f \in \mathrm{L}^2(\mathbb{R}_+);\ \mathbf{L}f \in \mathrm{L}^2(\mathbb{R}_+), f^+(0) = 0\}. \tag{2.67}$$

Moreover, for any $t \geq 0$, $Q_t \in \mathbf{B}\mathrm{L}^2(\mathbb{R}_+)$ with $S(Q_t) = S_c(Q_t) = (e^{-q^\alpha t})_{q\geq 0}$ and $S_p(Q_t) = S_r(Q_t) = \emptyset$. Finally, using the spectral expansion of the self-adjoint squared Bessel operator K_t, see e.g. [79, Section 6] and [78], one can deduce that for any $t > 0$ and $f \in \mathrm{L}^2(\mathbb{R}_+)$, $Q_t f$ has the following spectral expansion in $\mathrm{L}^2(\mathbb{R}_+)$,

$$Q_t f = H_\alpha \mathbf{e}_{\alpha,t} H_\alpha f. \tag{2.68}$$

CHAPTER 3

INTERTWINING, EXCURSION THEORY AND KREIN THEORY OF STRINGS FOR NON-SELF-ADJOINT MARKOV SEMIGROUPS

3.1 Introduction

The famous problem "Can we hear the shape of a drum?" raised by Kac [60] in 1966 has attracted much attention in the past decades. The question asks that whether one can determine a planar region $\Omega \subseteq \mathbb{R}^2$, up to geometric congruence, from the knowledge of all the eigenvalues of the problem

$$\frac{1}{2}\Delta u + \lambda u \quad = \quad 0 \qquad \text{on } \Omega,$$

where Δ is the Laplacian operator, with Dirichlet or Neumann boundary conditions. In other words, if we consider the triplet $(\Delta, \Omega, (\lambda_n)_{n\geq 0})$ where $(\lambda_n)_{n\geq 0}$ represents the sequence of eigenvalues of Δ on Ω, then Kac's problem asks if Ω can be determined by providing $(\lambda_n)_{n\geq 0}$. It was not until 1992 that Gordon, Webb and Wolpert [55] answered this question negatively by constructing a counterexample with two non-congruent planar domains Ω_1 and Ω_2 which are *isospectral*, that is, the sequence of eigenvalues of Δ on these domains coincide, counted with multiplicities. These domains are the first planar instances of non-isometric, isospectral, compact connected Riemannian manifolds that were previously enunciated by Sunada [111] in the context of the Laplace Beltrami operator. An equivalent formulation of Kac's problem can be described as follows. Writing $(P_t^{\Omega_j})_{t\geq 0}$, $j = 1, 2$, the semigroups generated by $\Delta|_{\Omega_j}$ on $\mathrm{L}^2(\Omega_j)$, and assuming that there exists a unitary operator $\Lambda : \mathrm{L}^2(\Omega_2) \mapsto \mathrm{L}^2(\Omega_1)$ such that

$$P_t^{\Omega_1}\Lambda f = \Lambda P_t^{\Omega_2} f$$

for all $f \in L^2(\Omega_2)$, then does it follow that Ω_1 and Ω_2 are congruent? This idea was first exploited by Bérard [9, 10] who reconsidered Sunada's isospectral problem by providing an explicit transplantation map, that is an intertwining operator which is an unitary isomorphism, which carries each eigenspace in $L^2(\Omega_2)$ into the corresponding eigenspace in $L^2(\Omega_1)$. In addition, Arendt [3] (resp. Arendt et al. [4]) showed that for subdomains of $\mathbb{R}^N$ (resp. for manifolds), if the intertwining operator is order isomorphic, that is, Λ is linear, bijective, and $f \geq 0$ *a.e.* $\Leftrightarrow \Lambda f \geq 0$ *a.e.*, then Ω_1 and Ω_2 are congruent, offering a positive answer to Kac's problem. Furthermore, Arendt et al. [5] considered a more general setting by studying isospectrality of the Dirichlet or Neumann type semigroups associated to elliptic operators, including non-self-adjoint ones, by means of the concept of similarity, which is an intertwining relationship with Λ a bounded operator with a bounded inverse from the Hilbert space $L^2(\Omega_1)$ to $L^2(\Omega_2)$. Note that the similarity relation between their corresponding semigroups is equivalent to the isospectral property in the case of Laplacians, but, in general, a stronger property for non-self-adjoint operators. On the other hand, for $\Omega_i \subset \mathbb{R}^2$, they also showed that it is impossible to have a similarity transform that simultaneously intertwins Dirichlet and Neumann operators on Ω_1 and Ω_2, and therefore there does not exist a similarity transform that intertwins elliptic operators with Robin boundary conditions.

In this paper, we reconsider these problems from another perspective. More specifically, we consider the intertwining relationship

$$P_t \Lambda f = \Lambda Q_t f \tag{3.1}$$

where $P = (P_t)_{t \geq 0}$ and $Q = (Q_t)_{t \geq 0}$ are two Markov semigroups defined on $L^2(\mathfrak{m}) = L^2(E, \mathfrak{m})$ and $L^2(m) = L^2(E, m)$, respectively, with $(E, \mathcal{E})$ a Lusin state space which contains a point $b \in E$ which is regular for the two semigroups,

$m, \mathfrak{m}$ two measures, and $\Lambda : \mathrm{L}^2(m) \mapsto \mathrm{L}^2(\mathfrak{m})$ is merely a densely defined closed and one-to-one operator. In other words, compared to Kac's framework, we are interested in a (weak) isospectrality from an analytical viewpoint rather than a geometric one: while the state space remains the same we consider different operators acting on this domain that intertwine in a weak sense. We emphasize that the fact that we do not require a similarity relation between the operators may imply that their spectrum differ drastically.

The first issue we investigate is to understand whether in our set up the intertwining relation is stable under any modification of the boundary conditions. For instance, is that possible that there exists an operator that links simultaneously the Dirichlet and Neumann operators, providing an opposite answer to the one obtained in [5] for identical operators acting on different planar domains? We shall show that indeed if two Dirichlet semigroups intertwin (in the sense given above) then any of their recurrent extensions in the sense of Itô, are also linked with the same operator. This includes for instance the case of Neumann boundary condition, but also reflecting type condition with a jump and sticky boundary conditions and a mixture of them. We carry on by providing sufficient conditions for the reverse claims to hold.

We proceed by studying the following question. Can one provide a probabilistic interpretation of intertwining relationships between Markov semigroups? This is a natural and fundamental question as this type of commutation relationships appears in various issues in recent studies of stochastic processes, see e.g. [92, 88, 91, 42, 49, 92]. We show that when two Dirichlet semigroups intertwin then any of its recurrent extension share, under an appropriate normalization, the same local time at the regular boundary point. Indeed we prove that the law of their inverse local time which is, from the general theory of Markov processes, a subordinator, is characterized by the same Bernstein function. This has the nice pathwise interpretation that the intertwining Markov processes behave the same at a common regular boundary point, but, of course, have different behavior elsewhere.

Next, we recall that the inverse local time of a quasi-diffusion also plays an important role in Krein's spectral theory of strings, since it contains information about the spectrum of the quasi-diffusion process killed at the boundary. Therefore, the question arises naturally that whether one can, through an intertwining relation with the semigroup of a quasi-diffusion, derive a similar result for non-diffusions. We answer this question positively by showing that if P and Q satisfy relation (3.1) with Q being the semigroup of a quasi-diffusion, then the Laplace exponent of the inverse local time of the (non-diffusion) Markov process corresponding to P also admits a Stieltjes representation, and the (densely defined) spectral measure of the killed semigroup of P is absolutely continuous with respect to the measure appearing in this Stieltjes representation. This defines a weaker version of Krein's property, which can be seen as an extension to Krein's theory to non-diffusions.

The rest of this paper is organized as follows. After this current section of introduction and basic setups, we start in Section 2 by stating our main theorem and its three corollaries, which give results on the intertwining of semigroups of recurrent extensions, excursion theory and Krein's theory of strings. We prove these results in Section 3. In Section 4, we provide two classes of semigroups which serve as examples for such intertwining relationship. In particular, we study the classes of positive self-similar semigroups and reflected generalized

Laguerre semigroups, and show that these (non-self-adjoint) semigroups intertwine with the Bessel semigroup and (classical) Laguerre semigroup respectively. We also deduce the expression for the Laplace exponents of their inverse local times. For a reflected generalized Laguerre semigroup, we also obtain in Section 4 its spectral expansion under some conditions, and derive its rate of convergence to equilibrium, which follows a perturbed spectral gap estimate.

3.1.1 Preliminaries

Let $(E, \mathcal{E})$ be a Lusin state space, and let $X = (X_t)_{t\geq 0}$ (resp. $Y = (Y_t)_{t\geq 0}$) defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t \geq 0, \mathbb{P})$ be a strong Markov process on E, which is assumed to have an infinite lifetime, and let $P = (P_t)_{t\geq 0}$ (resp. $Q = (Q_t)_{t\geq 0}$) denote its corresponding Borel right semigroup, that is, $P_t f(x) = \mathbb{E}_x[f(X_t)]$ (resp. $Q_t f(x) = \mathbb{E}_x[f(Y_t)]$) for $f \in \mathrm{B}_b(E)$, where $\mathbb{E}_x$ denote the expectation under measure $\mathbb{P}_x(X_0 = x) = 1$ (resp. $\mathbb{P}_x(Y_0 = x) = 1$). We also assume that for any $f \in \mathsf{C}_b(E)$ (resp. $\mathrm{B}_b(E)$) and $x \in E$, the mappings

$$t \mapsto P_t f(x) \text{ and } t \mapsto Q_t f(x) \text{ are continuous (resp. Borel),} \tag{3.2}$$

and we recall that condition (3.2) also means that P_t and Q_t are *stochastically continuous*, see e.g. [38, Definition 5.1]. We further suppose that $b \in E$ is a regular point for itself, that is $\mathbb{P}_b(T_b^X = 0) = \mathbb{P}_b(T_b^Y = 0) = 1$, where $T_b^X = \inf\{t > 0; X_t = b\}$ is the hitting time of b for process X, and T_b^Y is defined similarly. Let $X^\dagger = (X_t^\dagger)_{t\geq 0} = (X_t;\ 0 \leq t \leq T_b^X)$ be the process X killed when it hits b, after which it is sent to the cemetery point Δ, where we adopt the usual convention that a real-valued function f on E can be extended to Δ by $f(\Delta) = 0$. We also let $P^\dagger = (P_t^\dagger)_{t\geq 0}$ denote the semigroup of $X^\dagger$, i.e. $P_t^\dagger f = \mathbb{E}_x[f(X_t);\ t < T_b^X]$, and we define the process

$Y^\dagger = (Y_t^\dagger)_{t\geq 0}$ along with its semigroup $Q^\dagger = (Q_t^\dagger)_{t\geq 0}$ in a similar fashion. Next, let $U_q f = \int_0^\infty e^{-qt} P_t f dt$ and $U_q^\dagger f = \int_0^\infty e^{-qt} P_t^\dagger f dt$ be the resolvents of P and $P^\dagger$, respectively, and, V_q and $V_q^\dagger$ be the resolvents of Q and $Q^\dagger$.

We now assume that there exists an excessive measure $\mathfrak{m}$ (resp. m) on $(E, \mathcal{E})$ for the semigroup P (resp. Q), i.e. $\mathfrak{m}$ (resp. m) is a σ-finite measure and $\mathfrak{m}P_t \leq \mathfrak{m}$ (resp. $mQ_t \leq m$) for all $t > 0$, and in particular, when $\mathfrak{m}P_t = \mathfrak{m}$ (resp. $mQ_t = m$), $\mathfrak{m}$ (resp. m) is an invariant measure. Then a standard argument, see [38, Theorem 5.8], indicates that P extends uniquely into a strongly continuous semigroup on $\mathrm{L}^2(\mathfrak{m})$, which is the weighted Hilbert space

$$\mathrm{L}^2(\mathfrak{m}) = \{f : E \to \mathbb{R} \text{ measurable }; \|f\|_{\mathfrak{m}} = \int_E f^2(x)\mathfrak{m}(dx) < \infty\}$$

endowed with the norm $\|\cdot\|_{\mathfrak{m}}$ (when there is no confusion and for sake of simplicity, If $\mathfrak{m}$ is absolutely continuous, we also use $\mathfrak{m}$ to denote its density and write $\mathrm{L}^2(\mathfrak{m})$ the Hilbert space with weight $\mathfrak{m}(x)dx$.) Similarly, Q also admits a strongly continuous extension to $\mathrm{L}^2(m)$. Note that since $\mathfrak{m}P_t^\dagger \leq \mathfrak{m}P_t \leq \mathfrak{m}$, $\mathfrak{m}$ is also an excessive measure for $P^\dagger$, hence $P^\dagger$ can also be uniquely extended to a strongly continuous semigroup on $\mathrm{L}^2(\mathfrak{m})$. Similar results holds for $Q^\dagger$ as well.

Now let us follow the construction as described in [54] to observe that there exists a left-continuous $\widehat{X} = (\widehat{X}_t)_{t\geq 0}$ under probabilty measure $\widehat{\mathbb{P}}_x$, which is the dual process of X with respect to $\mathfrak{m}$, and is moderate Markov. Note that the measures $(\widehat{\mathbb{P}}_x)_{x\in E}$ are only determined modulo an $\mathfrak{m}$-polar set. Let $\widehat{P}_t f = \widehat{\mathbb{P}}_x[f(\widehat{X}_t)]$ denote the moderate Markov dual semigroup associated with $\widehat{X}$ and $\widehat{U}_q$ be the resolvent, then $\widehat{P}$ and $\widehat{U}_q$ are linked to P and U_q via the duality formula

$$(P_t f, g)_{\mathfrak{m}} = (f, \widehat{P}_t g)_{\mathfrak{m}}, \quad (U_q f, g)_{\mathfrak{m}} = (f, \widehat{U}_q g)_{\mathfrak{m}}$$

for each $f, g \in \mathrm{B}_b(\mathbb{R}_+)(E), q > 0, t \geq 0$, where we recall that

$$(f, g)_{\mathfrak{m}} = \int_E f(x)g(x)\mathfrak{m}(dx) \tag{3.3}$$

whenever this integral exists.

Because b is a regular point, the singleton $\{b\}$ is not semipolar and there exists a local time $\mathfrak{l}^X$ at b, which is a positive continuous additive functional of X, increasing only on the visiting set $\{t \geq 0;\ X_t = b\}$. We mention that $\mathfrak{l}^X$ is uniquely determined up to a multiplicative constant. The inverse local time $\tau^X = (\tau^X_t)_{t\geq 0}$ is the right continuous inverse of $\mathfrak{l}^X$, i.e.

$$\tau^X_t = \inf\{s > 0;\ \mathfrak{l}^X_s > t\}, \quad t \geq 0.$$

It is a standard argument that under the law $\mathbb{P}_x$, τ^X is a strictly increasing subordinator and therefore for any $q > 0$,

$$\mathbb{E}_x[e^{-q\tau^X_t}] = e^{-t\Phi_X(q)},$$

where $\Phi_X(q)$ is the Laplace exponent of τ^X and admits the following Lévy-Khintchin representation

$$\Phi_X(q) = \delta_X + q\gamma_X + \int_0^\infty (1 - e^{-qr})\mu_X(dr), \tag{3.4}$$

with $\delta_X = \lim_{q\to 0} \Phi_X(q)$ is the so-called killing parameter, $\gamma_X = \lim_{q\to\infty} \frac{\Phi_X(q)}{q}$ is the so-called elasticity parameter, and μ_X is the Lévy measure of τ^X, that is a σ-finite measure concentrated on $(0, \infty)$ satisfying $\int_0^\infty (1 \wedge y)\mu_X(dy) < \infty$. Furthermore, we follow [100, Chapter X, Section 2] to define the so-called Revuz measure $\mathfrak{R}_{\mathfrak{l}^X}$ for local time $\mathfrak{l}^X$ as

$$\mathfrak{R}_{\mathfrak{l}^X} f = \lim_{t\to 0} \frac{1}{t}\mathbb{E}_{\mathfrak{m}}\left[\int_0^t f(X_s)d\mathfrak{l}^X_s\right],$$

which, in the case when $\mathfrak{m}$ is an invariant measure, can be defined by the simpler formula

$$\mathfrak{R}_{\mathfrak{l}^X} f = \mathbb{E}_{\mathfrak{m}}\left[\int_0^1 f(X_s)d\mathfrak{l}^X_s\right].$$

Its total mass, denoted by $c(\mathfrak{m})$, is

$$c(\mathfrak{m}) = \mathfrak{R}_{\mathfrak{l}^X}\mathbf{1}, \tag{3.5}$$

which is a positive constant. *Since the local time can be defined up to a multiplicative constant, in order to streamline the discussion, we suppose for the remainder of this paper that the local time $\mathfrak{l}^X$ has been normalized so that $c(\mathfrak{m}) = 1$.* The notations for $\mathfrak{l}^Y, \tau^Y, \Phi_Y(q), \delta_Y, \gamma_Y, \mu_Y$ are trivial to understand, and we also suppose that $\mathfrak{l}^Y$ has been normalized to make $c(m) = 1$.

Moreover, by [51, Proposition (A.4)], since b is regular, we have $\widehat{\mathbb{P}}_b[T_b^{\widehat{X}} = 0] = 1$, where $T_b^{\widehat{X}}$ is hitting time of $\widehat{X}$ to b. Let $\widehat{X}^\dagger = (\widehat{X}_t)_{t<T_b^{\widehat{X}}}$ denote the process $\widehat{X}$ killed at b, and $\widehat{P}^\dagger$ and $\widehat{U}_q^\dagger$ for its semigroup and resolvent. In addtion, for $x \in E$, we let

$$\varphi_q^X(x) = \mathbb{E}_x[e^{-qT_b^X}], \varphi^X(x) = \varphi_0^X(x) = \mathbb{P}_x[T_b^X < \infty], \varphi_q^{\widehat{X}}(x) = \mathbb{E}_x[e^{-qT_b^{\widehat{X}}}], \varphi^{\widehat{X}}(x) = \varphi_0^{\widehat{X}}(x).$$

It is well-known that strong Markov property implies the following relation, for any $x \in E$ and $f \in \mathrm{B}_b(E) \cup \mathrm{L}^2(\mathfrak{m})$,

$$U_q f(x) = U_q^\dagger f(x) + \varphi_q^X(x) U_q f(b). \tag{3.6}$$

On the other hand, although the dual process $\widehat{X}$ is moderate Markov, by [51, Corollary (A.11)], we have for all $f \in \mathrm{B}_b^+(E)$,

$$\widehat{U}_q f(x) = \widehat{U}_q^\dagger f(x) + \varphi_q^{\widehat{X}}(x) \widehat{U}_q f(b). \tag{3.7}$$

Similarly there exists a moderate Markov dual process $\widehat{Y}$ associated with Y and m, whose semigroup and resolvent are denoted by $\widehat{Q}$ and $\widehat{V}_q$ respectively. The killed process is denoted by $\widehat{Y}^\dagger$ and its semigroup and resolvent are denoted by $\widehat{Q}^\dagger$ and $\widehat{V}_q^\dagger$, and the notations $\varphi_q^Y, \varphi^Y, \varphi_q^{\widehat{Y}}, \varphi^{\widehat{Y}}$ are self-explanatory.

3.2 Statements of main results

In this section, we will state the main theorem and some of its corollaries. We start by defining a few notations. For two sets A and B, we write $A \subseteq_d B$ if $A \subseteq B$ and $\overline{A} = B$, where $\overline{A}$ is the closure of A. Moreover, for some operator Λ, we denote $\mathcal{D}_\Lambda$ to be its domain, and we define the following class of operators

$$C(m, \mathfrak{m}) = \{\Lambda : \mathcal{D}_\Lambda \subseteq_d \mathrm{L}^2(m) \to \mathrm{Ran}(\Lambda) \subseteq_d \mathrm{L}^2(\mathfrak{m}) \text{ linear, injective and closed.}\}. \tag{3.8}$$

Note that if $\Lambda \in C(m, \mathfrak{m})$, then $\widehat{\Lambda} \in C(\mathfrak{m}, m)$ where $\widehat{\Lambda}$ is the L^2-adjoint of Λ, i.e. for any $f \in \mathcal{D}_\Lambda, g \in \mathcal{D}_{\widehat{\Lambda}}$, we have $\langle \Lambda f, g \rangle_\mathfrak{m} = \left\langle f, \widehat{\Lambda} g \right\rangle_m$, where $\langle \cdot, \cdot \rangle_\mathfrak{m}$ (resp. $\langle \cdot, \cdot \rangle_m$) denotes the standard inner product in $\mathrm{L}^2(\mathfrak{m})$ (resp. $\mathrm{L}^2(m)$). In addition, we say Λ is *mass preserving* if $\Lambda \mathbf{1}_E \equiv \mathbf{1}_E$ where $\mathbf{1}_E(x) = 1$ for all $x \in E$, and it is assumed that $\mathbf{1}_E$ is in the (possibly) extended domain of Λ. Then we have the following results.

3.2.1 Intertwining relations and inverse local time

The main result of this section is stated in the following Theorem.

Theorem 3.2.1. *Let $\Lambda \in C(m, \mathfrak{m})$, with both Λ and $\widehat{\Lambda}$ being mass preserving. Consider the following claims.*

1. $P_t^\dagger \Lambda f = \Lambda Q_t^\dagger f$ *for all* $f \in \mathcal{D}_\Lambda \cup \{\mathbf{1}_E\}$.
2. $P_t \Lambda f = \Lambda Q_t f$ *for all* $f \in \mathcal{D}_\Lambda \cup \{\mathbf{1}_E\}$.
3. *For any $q > 0$, we have $\varphi_q^X(x) = \Lambda \varphi_q^Y(x)$ $\mathfrak{m}$-almost everywhere (a.e. for short) on E, and $\varphi_q^{\widehat{Y}}(x) = \Lambda \varphi_q^{\widehat{X}}(x)$ m-a.e. on E.*

4. $\Phi_X(q) = \Phi_Y(q)$ *for each* $q > 0$.

Then, we have

$$(1) \Rightarrow (3) \Rightarrow (4) \text{ and } (1) \Rightarrow (2).$$

If in addition, writing $\mathbf{1}_{\{b\}}$ *the indicator function at* $\{b\}$*, we have*

$$\begin{aligned} &\Lambda \mathbf{1}_{\{b\}}(x) = \mathbf{1}_{\{b\}}(x), \quad \widehat{\Lambda}\mathbf{1}_{\{b\}}(x) = \mathbf{1}_{\{b\}}(x) \text{ for any } x \in E, \text{ and} \\ &\Lambda Q_t f(b) = Q_t f(b), \quad \widehat{\Lambda}\widehat{P}_t g(b) = \widehat{P}_t g(b) \text{ for all } f \in \mathcal{D}_\Lambda \cup \{\mathbf{1}_E\}, g \in \mathcal{D}_{\widehat{\Lambda}} \cup \{\mathbf{1}_E\}, \end{aligned} \tag{3.9}$$

then

$$(2) \Rightarrow (3) \text{ and } (1) \Leftrightarrow (2).$$

Remark 3.2.1. *1. Note that* Λ *can be defined up to a multiplicative constant* c*, hence the mass preserving condition (resp. condition* (3.9)*) can be stated in a slightly more general way as, there exists a constant* $c \neq 0$ *such that* $c\Lambda$ *is mass preserving (resp. satisfies* (3.9)*).*

2. *If* m *is of finite mass on* E*, then clearly* $\mathbf{1}_E \in \mathrm{L}^2(m)$*. Otherwise, we understand the conditions* (1) *and* (2) *as* Q_t *and* P_t *acting as a Markov operator on* $\mathrm{B}_b(\mathbb{R}_+)(E)$*. For sake of simplicity, we keep the same notations as the* L^2*-semigroups.*

Corollary 3.2.1. *Under assumption* (1) *or equivalently,* (2) *together with the additional condition* (3.9) *for* Λ*, then* Λ *also intertwins two generators with Robin boundary condition at* b*.*

Here we address that as opposed to the setting in [5], where there are no similarity transforms between two Laplacians acting on two isospectral domains with Robin boundary condition, our situation is different in two aspects. First, the two generators are acting on the same space and both have the same boundary at 0. Second, the intertwining operator Λ that we consider in this paper is not a similarity transform as in [5]. Therefore, we see that under a different setting, there indeed exists an intertwining relation between two Robin type generators.

3.2.2 Excursion theory

We now provide a further probabilistic explanation for the intertwining relation by means of excursion theory. We first recall from Maisonneuve [72] that, for the excursions of X from the regular point b, we can associate an exit system $(\mathbf{P}, l^X)$ where $\mathbf{P}$ is the so-called (Maisonneuve) excursion measure. Moreover, let us define the collection of σ-finite measures $(\mathbf{P}_t)_{t>0}$ by

$$\mathbf{P}_t(f) = \mathbf{P}[f(X_t), t < T_b],$$

for any $f \in \mathrm{B}_b^+(E)$. Then $(\mathbf{P}_t)_{t>0}$ is an entrance law of semigroup $P^\dagger$, in other words, $\mathbf{P}_{s+t} = \mathbf{P}_s P_t^\dagger$ for any $s > 0, t \geq 0$. Furthermore, for any $q > 0$, we define $\mathbf{U}_q(f) = \int_0^\infty e^{-qt}\mathbf{P}_t(f)dt$. Similarly, let $\mathbf{Q}$ denote the Maisonneuve excursion measure for process Y, $(\mathbf{Q}_t)_{t>0}$ be the associated entrance law, and $\mathbf{V}_q(f) = \int_0^\infty e^{-qt}\mathbf{Q}_t(f)dt$. We use $l_X(a)$ (resp. $l_Y(a)$) to denote the length of the first excursion interval with length $l > a$ for the process X (resp. Y). In addition, we let M_X (resp. M_Y) denote the closure in $[0, \infty)$ of the visiting set $\{t \geq 0; X_t = b\}$ (resp. $\{t \geq 0; Y_t = b\}$), and $\zeta_X = \sup M_X$ (resp. $\zeta_Y = \sup M_Y$) be the last exit time of X (resp. Y) from b. Then we have the following corollary.

Corollary 3.2.2. *Under the assumption in Theorem 3.2.1* (1)*, the following statements hold.*

(a) For any $A \in \mathcal{B}(\mathbb{R}^+)$ a Borel set, we have $\mathbf{P}(T_b^X \in A) = \mathbf{Q}(T_b^Y \in A)$.

(b) For every $a \in \mathbb{R}_+$, $l_X(a)$ and $l_Y(a)$ have the same distribution.

(c) For every $x > 0$, ζ_X and ζ_Y have the same distribution under $\mathbb{P}_x$.

3.2.3 Krein's spectral theory of strings

We first recall that the Laplace exponent of the inverse local time is an essential object in Krein's spectral theory of strings, for which we will provide a brief review of the known results herein, and we refer to [65, 63] for an excellent account. For sake of simplicity, here we take $b = 0$ as the regular boundary but note that the choice of 0 is indeed arbitrary. Suppose Y is the Markov process corresponding to the generalized second order differential operator $\mathbf{G} = \frac{d}{d\mathtt{m}}\frac{d}{dx}$ with boundary condition $f^-(0) = \lim_{x\downarrow 0}\frac{f(0)-f(-x)}{x} = 0$, where $\mathtt{m}$ is a string, that is a right-continuous and non-decreasing function defined on $[0, l) \to [0, \infty)$ for some $0 < l = l(\mathtt{m}) \leq \infty$ with $\mathtt{m}(0) = 0$. Then Y is called a quasi-diffusion (also called generalized diffusion or gap diffusion) with 0 being a regular boundary. In this case, it is known that Φ_Y is a Pick function, that is, a holomorphic function that preserves the upper half-plane, i.e. $\Im(\Phi_Y(z)) \geq 0$ for all $\Im(z) > 0$. Moreover, recalling the Lévy-Khintchin representation of Φ_Y as given in (3.4), then the Lévy measure μ_Y admits a density u_Y which is completely monotone, with

$$u_Y(r) = \int_0^\infty e^{-rq}\nu_Y(dq), \tag{3.10}$$

for some ν_Y a measure satisfying $\int_0^\infty \frac{\nu_Y(dq)}{1+q} < \infty$, and $\delta_Y = \nu_Y(\{0\})$.

Indeed, let $\mathfrak{M}$ and $\mathfrak{P}$ denote the spaces of strings and Pick functions, respectively, then Krein's theory shows that there exists a bijection between $\mathfrak{M}$ and $\mathfrak{P}$, in the sense that for any Pick function $\Phi \in \mathfrak{P}$, there exists a quasi-diffusion Y with generator $\frac{d}{d\mathtt{m}}\frac{d}{dx}$ for some $\mathtt{m} \in \mathfrak{M}$, such that Φ is the Laplace exponent of the inverse local time of Y. The converse also holds. Moreover, recalling that $Q_t^\dagger$ is the semigroup of Y killed at hitting 0, and let $\mathbf{G}^\dagger$ denote its infinitesimal generator, defined as

$$\mathbf{G}^\dagger f = \lim_{t\to 0} \frac{Q_t^\dagger f - f}{t}$$

for f in domain $\mathcal{D}(\mathbf{G}^\dagger) = \{f \in \mathrm{L}^2(m);\ \mathbf{G}^\dagger f \in \mathrm{L}^2(m)\}$. We also recall that a family of orthogonal projection operators $\mathtt{E} = (\mathtt{E}_q)_{q\in(-\infty,\infty)}$ on $\mathrm{L}^2(m)$ is called a resolution of identity if for all $f \in \mathrm{L}^2(m)$,

1. $\lim_{q\uparrow r} \mathtt{E}_q f = \mathtt{E}_r f$, i.e. $\mathtt{E}_q$ is strongly left continuous for all $q \in (-\infty, \infty)$.
2. $\lim_{q\downarrow -\infty} \mathtt{E}_q f = 0, \lim_{q\uparrow\infty} \mathtt{E}_q f = f$.
3. $\mathtt{E}_q \mathtt{E}_r f = \mathtt{E}_{\min(q,r)} f$.

Note that since $\mathbf{G}^\dagger$ is a self-adjoint operator, it generates a unique resolution of identity $\mathtt{E}^Y = (\mathtt{E}_q^Y)_{q\in(-\infty,\infty)}$, which can be represented by

$$\mathtt{E}_q^Y = \mathbf{1}_{(-\infty,q]}(\mathbf{G}^\dagger). \tag{3.11}$$

Finally, let $\sigma(\mathbf{G}^\dagger)$ represent the spectrum of $\mathbf{G}^\dagger$, then Y (or its corresponding semigroup Q) satisfies *the Krein's property*, which is defined as follows.

1. For any $f \in \mathrm{L}^2(m)$, $Q_t^\dagger f$ admits the spectral expansion in $\mathrm{L}^2(m)$
$$Q_t^\dagger f(x) = \int_{\sigma(\mathbf{G}^\dagger)} e^{-qt} d\mathtt{E}_q^Y f. \tag{3.12}$$
2. For any $f, g \in \mathrm{L}^2(m)$, the signed measure $\left\langle d\mathtt{E}_q^Y f, g\right\rangle_m$ is absolutely continuous with respect to $\nu_Y(dq)$, the spectral measure of the Pick function Φ_Y as shown in (3.4) and (3.10), and the Radon-Nikodym derivative between these two measures is given by
$$\frac{\left\langle d\mathtt{E}_q^Y f, g\right\rangle_m}{\nu_Y(dq)} = (f, h_q)_m (g, h_q)_m \tag{3.13}$$

for some function h_q.

During the last decades, there have been a lot of nice developments of Krein's theory of strings, see e.g. Kotani [62] for a generalization of Krein's theory into the case of singular boundaries. However, these works are still in the framework of quasi-diffusion or differential operator. In what follows, we propose an extension of Krein's theory to general Markov semigroups. Since these linear operators are in general non-self-adjoint operators (neither normal), meaning that there is no spectral theorem available, we need to introduce this weaker notion of resolution of identity. First, fix some interval $[\alpha,\beta]$, $-\infty \leq \alpha < \beta \leq \infty$, we follow [27] to define a *non-self-adjoint resolution of identity* as a family of measure-valued operators $\mathtt{E} = (\mathtt{E}_q)_{q\in[\alpha,\beta]} : \mathcal{D}(\mathtt{E}) \to \mathrm{L}^2(\mathfrak{m})$ which satisfies the following.

(i) $\mathcal{D}(\mathtt{E}) \subseteq_d \mathrm{L}^2(\mathfrak{m})$ and $\mathtt{E}_q\mathcal{D}(\mathtt{E}) \subseteq \mathcal{D}(\mathtt{E})$ for all $q \in [\alpha,\beta]$.

(ii) $\mathtt{E}_\alpha f = 0, \mathtt{E}_\beta f = f$ for all $f \in \mathcal{D}(\mathtt{E})$.

(iii) $\mathtt{E}_q\mathtt{E}_r f = \mathtt{E}_{\min(q,r)} f$ for all $q, r \in [\alpha,\beta], f \in \mathcal{D}(\mathtt{E})$.

Definition 3.2.3.1. *Suppose that* $\{0\}$ *is a regular point for* X*, then we say* X *(or its corresponding semigroup* P*) satisfies the weak-Krein property if the following conditions hold.*

(i) The Lévy measure μ_X *of* Φ_X *(the Laplace exponent of the inverse local time at 0) has a completely monotone density, which can be represented in the form* (3.10) *for some measure* ν_X*.*

(ii) There exists a Borel set C *and* $\mathcal{D}(\mathtt{E}^X) \subseteq_d \mathrm{L}^2(\mathfrak{m})$ *such that on* $\mathcal{D}(\mathtt{E}^X)$*,*

$$P_t^\dagger = \int_C e^{-qt} d\mathtt{E}_q^X \tag{3.14}$$

for any $t > 0$, where $\mathtt{E}^X = (\mathtt{E}^X_q)_{q\in[\inf C, \sup C]}$ is a non-self-adjoint resolution of identity on $\mathcal{D}(\mathtt{E}^X)$.

(iii) $\left\langle d\mathtt{E}^X_q f, g\right\rangle_{\mathfrak{m}}$ *is absolutely continuous with respect to ν_X for any $f \in \mathcal{D}(\mathtt{E}^X), g \in$* $\mathrm{L}^2(\mathfrak{m})$.

Note that the weak-Krein property only requires the spectral expansion (3.14) to hold on a dense subset of $\mathrm{L}^2(\mathfrak{m})$, which is distinguished from the Krein property for quasi-diffusions, where this expansion holds on the entire Hilbert space. Then we have the following corollary.

Corollary 3.2.3. *Suppose that Theorem 3.2.1*(1) *holds, with Q being the semigroup of a quasi-diffusion and $\Lambda \in \mathbf{B}(\mathrm{L}^2(m), \mathrm{L}^2(\mathfrak{m}))$. Further assume that for any $q \in \sigma(\mathbf{G}^\dagger)$, $\mathtt{E}^Y_q g \in \mathcal{D}_\Lambda$ for all $g \in \mathcal{D}_\Lambda$, then P has the weak-Krein property, with $C = \sigma(\mathbf{G}^\dagger)$.*

3.3 Proof of Theorem 3.2.1 and its corollaries

3.3.1 Proof of Theorem 3.2.1

We start the proof with the following results, which may of independent interest.

Lemma 3.3.1. *Assume that* (1) *(resp.* (2)*) holds, then for any $f \in \mathcal{D}_\Lambda$ and $q > 0$, we have*

$$U^\dagger_q \Lambda f = \Lambda V^\dagger_q f. \tag{3.15}$$

$$(\text{resp. } U_q \Lambda f = \Lambda V_q f.) \tag{3.16}$$

Proof. First, assuming that (2) holds and let us define for any $n > 0$, $U_q^n f = \int_0^n e^{-qt} P_t f dt$ and $V_q^n f = \int_0^n e^{-qt} Q_t f dt$, then by the intertwining relation, we have, for $f \in \mathcal{D}_\Lambda$,

$$U_q^n \Lambda f = \int_0^n e^{-qt} P_t \Lambda f dt = \int_0^n e^{-qt} \Lambda Q_t f dt = \Lambda \int_0^n e^{-qt} Q_t f dt = \Lambda V_q^n f.$$

However, note that $\lim_{n\to\infty} V_q^n f = V_q f$ in $\mathrm{L}^2(m)$, and $\lim_{n\to\infty} \Lambda V_q^n f = \lim_{n\to\infty} U_q^n \Lambda f = U_q \Lambda f$ in $\mathrm{L}^2(\mathfrak{m})$, then by the closeness property of Λ, we have

$$\Lambda V_q f = U_q \Lambda f.$$

Similar arguments hold under assumption (1) and this completes the proof. □

Lemma 3.3.2. *For each $q > 0$, we have $\varphi_q^X, \varphi_q^{\widehat{X}} \in \mathrm{L}^2(\mathfrak{m})$ and $\varphi_q^Y, \varphi_q^{\widehat{Y}} \in \mathrm{L}^2(m)$.*

Proof. First, by [51, Theorem (3.6)(ii)], we can write

$$(\mathbf{1}_E, \varphi_q^{\widehat{X}})_{\mathfrak{m}} = (\delta_X + q(\varphi_q^X, \varphi^{\widehat{X}})_{\mathfrak{m}}) U_q \mathbf{1}_E(b).$$

Now since $qU_q\mathbf{1}_E(b) \leq 1$ and $\delta_X + q(\varphi_q^X, \varphi^{\widehat{X}})_{\mathfrak{m}} = \Phi_X(q) < \infty$, we see that $(\mathbf{1}_E, \varphi_q^{\widehat{X}})_{\mathfrak{m}} < \infty$ for each $q > 0$, i.e. $\varphi_q^{\widehat{X}} \in \mathrm{L}^1(\mathfrak{m})$ since it is non-negative. Moreover, since $\varphi_q^{\widehat{X}}(x) \leq 1$ for all x, we have

$$\int_0^\infty \left(\varphi_q^{\widehat{X}}(x)\right)^2 \mathfrak{m}(dx) \leq \int_0^\infty \varphi_q^{\widehat{X}}(x)\mathfrak{m}(dx) = (\mathbf{1}_E, \varphi_q^{\widehat{X}})_{\mathfrak{m}} < \infty.$$

Therefore $\varphi_q^{\widehat{X}} \in \mathrm{L}^2(\mathfrak{m})$. Similarly, we have

$$(\mathbf{1}_E, \varphi_q^X)_{\mathfrak{m}} = (\delta_X + q(\varphi_q^{\widehat{X}}, \varphi^X)_{\mathfrak{m}}) \widehat{U}_q \mathbf{1}_E(b).$$

By [51, Proposition (3.9)], $\delta_X + q(\varphi_q^{\widehat{X}}, \varphi^X)_{\mathfrak{m}} = \delta_X + q(\varphi_q^X, \varphi^{\widehat{X}})_{\mathfrak{m}} < \infty$, while on the other hand $q\widehat{U}_q\mathbf{1}_E(b) \leq 1$, hence $\varphi_q^X \in \mathrm{L}^1(\mathfrak{m})$ and also in $\mathrm{L}^2(\mathfrak{m})$ since it is bounded by 1. The same arguments apply for the proof for φ_q^Y and $\varphi_q^{\widehat{Y}}$, and this completes the proof of this Lemma. □

Proof of (1) $\Rightarrow$ (3)

Note that for any $x \in E_\dagger$ where we denote $E_b = E\backslash\{b\}$, we have $\mathbb{P}_x(T_b^X = 0) = 0$, hence since X has an a.s. infinite lifetime, we can rewrite $\varphi_q^X(x)$ using integration by parts, which yields

$$\begin{aligned}\varphi_q^X(x) &= \int_0^\infty e^{-qt}\mathbb{P}_x(T_b^X \in dt) = \int_0^\infty qe^{-qt}\mathbb{P}_x(T_b^X \leq t)dt = 1 - \int_0^\infty qe^{-qt}P_t^\dagger \mathbf{1}_E(x)dt \\ &= 1 - qU_q^\dagger \mathbf{1}_E(x),\end{aligned}$$

where we used the fact that $P_t^\dagger \mathbf{1}_E(x) = \mathbb{P}_x(T_b^X > t)$. On the other hand, since b is regular for itself, we have $\varphi_q^X(b) = 1$. Combining with the fact that $U_q^\dagger \mathbf{1}_E(b) = 0$, we see that for all $x \in E$,

$$\varphi_q^X(x) = (\mathbf{1}_E - qU_q^\dagger \mathbf{1}_E)(x). \tag{3.17}$$

Similarly, we have $\varphi_q^Y(x) = (\mathbf{1}_E - qV_q^\dagger \mathbf{1}_E)(x)$. Furthermore, by recalling that $\Lambda \mathbf{1}_E = \mathbf{1}_E$ and applying Lemma 3.3.1, we have

$$U_q^\dagger \mathbf{1}_E(x) = U_q^\dagger \Lambda \mathbf{1}_E(x) = \Lambda V_q^\dagger \mathbf{1}_E(x). \tag{3.18}$$

Combining the above results, we get that for any $q > 0$ and $x \in E$,

$$\varphi_q^X(x) \;\; = \;\; (\mathbf{1}_E - qU_q^\dagger \mathbf{1}_E)(x) = \Lambda\left(\mathbf{1}_E - qV_q^\dagger \mathbf{1}_E\right)(x) = \Lambda\varphi_q^Y(x).$$

Since we have shown $\varphi_q^Y \in \mathrm{L}^2(m)$, we also see that $\varphi_q^Y \in \mathcal{D}_\Lambda$. Next, by (1), we deduce easily the series of identities that for any $f \in \mathcal{D}_\Lambda, g \in \mathcal{D}_{\widehat{\Lambda}}$,

$$\left\langle f, \widehat{\Lambda}\widehat{P}_t^\dagger g\right\rangle_m = \left\langle \Lambda f, \widehat{P}_t^\dagger g\right\rangle_{\mathfrak{m}} = \left\langle P_t^\dagger \Lambda f, g\right\rangle_{\mathfrak{m}} = \left\langle \Lambda Q_t^\dagger f, g\right\rangle_{\mathfrak{m}} = \left\langle Q_t^\dagger f, \widehat{\Lambda} g\right\rangle_{\mathfrak{m}} = \left\langle f, \widehat{Q}_t^\dagger \widehat{\Lambda} g\right\rangle_{\mathfrak{m}}, \tag{3.19}$$

which means that $\widehat{Q}_t^\dagger \widehat{\Lambda} g - \widehat{\Lambda}\widehat{P}_t^\dagger g \in \mathcal{D}_{\widehat{\Lambda}}^\perp = \{0\}$ since $\overline{\mathcal{D}_{\widehat{\Lambda}}} = \mathrm{L}^2(\mathfrak{m})$. Therefore, $\widehat{P}^\dagger$ and $\widehat{Q}^\dagger$ have the intertwining relation on $\mathcal{D}_{\widehat{\Lambda}}$,

$$\widehat{\Lambda}\widehat{P}_t^\dagger = \widehat{Q}_t^\dagger \widehat{\Lambda}.$$

By [51, Proposition (A.6)], we have $\widehat{\mathbb{P}}_y(T_b^{\widehat{Y}} = 0) = 0$ for all $y \in E_b \backslash S$ where S is an m-semipolar set, which m does not charge. On the other hand, since we are assuming that $\widehat{\Lambda}$ is also mass preserving, we can use the same arguments as above to prove that $\widehat{\Lambda}\varphi_q^{\widehat{X}}(x) = \varphi_q^{\widehat{Y}}(x)$ for all $q > 0$ and $x \in E_b \backslash S$. This completes the proof.

Proof of (3) $\Rightarrow$ (4)

Recall from [51, Theorem 3.6] that under the normalization $c(\mathfrak{m}) = 1$, the Laplace exponent of the inverse local time can be written as

$$\Phi_X(q) = \delta_X + q(\varphi_q^X, \varphi^{\widehat{X}})_{\mathfrak{m}}, \tag{3.20}$$

where we recall that the notation $(\cdot, \cdot)_{\mathfrak{m}}$ is given in (1.2), which means that $(\varphi_q^X, \varphi^{\widehat{X}})_{\mathfrak{m}} < \infty$ for all $q > 0$. Similarly, $(\varphi_q^Y, \varphi^{\widehat{Y}})_m < 0$ for all $q > 0$. On the other hand, by Lemma 3.3.2, we see that $\varphi_q^X, \varphi_q^{\widehat{X}} \in \mathrm{L}^2(\mathfrak{m})$ and $\varphi_q^Y, \varphi_q^{\widehat{Y}} \in \mathrm{L}^2(m)$ for any $q > 0$. Hence the assumption (3) implies that for any $q, r > 0$,

$$\left\langle \varphi_q^X, \varphi_r^{\widehat{X}} \right\rangle_{\mathfrak{m}} = \left\langle \Lambda\varphi_q^Y, \varphi_r^{\widehat{X}} \right\rangle_{\mathfrak{m}} = \left\langle \varphi_q^Y, \widehat{\Lambda}\varphi_r^{\widehat{X}} \right\rangle m = \left\langle \varphi_q^Y, \varphi_r^{\widehat{Y}} \right\rangle_m .$$

Next, since plainly $\varphi_r^{\widehat{X}}(x) \uparrow \varphi^{\widehat{X}}(x)$ and $\varphi_r^{\widehat{Y}}(x) \uparrow \varphi^{\widehat{Y}}(x)$ pointwise as $r \downarrow 0$, we easily deduce by monotone convergence that

$$(\varphi_q^X, \varphi^{\widehat{X}})_{\mathfrak{m}} = \lim_{r \downarrow 0}(\varphi_q^X, \varphi_r^{\widehat{X}})_{\mathfrak{m}} = \lim_{r \downarrow 0} \left\langle \varphi_q^X, \varphi_r^{\widehat{X}} \right\rangle_{\mathfrak{m}} = \lim_{r \downarrow 0} \left\langle \varphi_q^Y, \varphi_r^{\widehat{Y}} \right\rangle_m = \lim_{r \downarrow 0}(\varphi_q^Y, \varphi_r^{\widehat{Y}})_m = (\varphi_q^Y, \varphi^{\widehat{Y}})_m,$$

where we used the fact that $(f, g)_m = \langle f, g \rangle_m$ for any $f, g \in \mathrm{L}^2(m)$. Moreover, from [51, Remark (3.21)], the killing term δ_X can be represented as

$$\begin{aligned} \delta_X &= \lim_{q \to \infty}(\varphi_q^{\widehat{X}}, \mathbf{1}_E - \varphi^X)_{\mathfrak{m}} = \lim_{q \to \infty}(\varphi_q^{\widehat{X}}, \Lambda(\mathbf{1}_E - \varphi^Y))_{\mathfrak{m}} \\ &= \lim_{q \to \infty}(\widehat{\Lambda}\varphi_q^{\widehat{X}}, \mathbf{1}_E - \varphi^Y)_m = \lim_{q \to \infty}(\varphi_q^{\widehat{Y}}, \mathbf{1}_E - \varphi^Y)_m = \delta_Y. \end{aligned}$$

Therefore, combining the above results yields

$$\Phi_X(q) = \delta_X + q(\varphi_q^X, \varphi^{\widehat{X}})_{\mathfrak{m}} = \delta_Y + q(\varphi_q^Y, \varphi^{\widehat{Y}})_m = \Phi_Y(q),$$

where we consider again the normalization $c(\mathfrak{m}) = c(m) = 1$. This finishes the proof of (3) $\Rightarrow$ (4).

Proof of (1) $\Rightarrow$ (2)

By [51, Theorem 3.6 (ii)], for any $f \in \mathrm{L}^2(\mathfrak{m})$ and $q > 0$, $U_q f(b)$ can be written as

$$U_q f(b) = \frac{(f, \varphi_q^{\widehat{X}})_{\mathfrak{m}}}{\Phi_X(q)} = \frac{\left\langle f, \varphi_q^{\widehat{X}} \right\rangle_{\mathfrak{m}}}{\Phi_X(q)},$$

where the second identity comes from Lemma 3.3.2. Since we have proved (1) $\Rightarrow$ (4) (resp. (1) $\Rightarrow$ (3)), which means that $\Phi_X = \Phi_Y$ (resp. $\widehat{\Lambda}\varphi_q^{\widehat{X}} = \varphi_q^{\widehat{Y}}$ m-a.e.), we deduce that, for $f \in \mathcal{D}_\Lambda$,

$$U_q \Lambda f(b) = \frac{\left\langle \Lambda f, \varphi_q^{\widehat{X}} \right\rangle_{\mathfrak{m}}}{\Phi_X(q)} = \frac{\left\langle f, \widehat{\Lambda}\varphi_q^{\widehat{X}} \right\rangle_m}{\Phi_Y(q)} = \frac{\left\langle f, \varphi_q^{\widehat{Y}} \right\rangle_m}{\Phi_Y(q)} = V_q f(b). \tag{3.21}$$

Furthermore, by (1), we have $U_q^\dagger \Lambda f = \Lambda V_q^\dagger f$, hence the strong Markov property (3.6) yields that for any $x \in E_b$,

$$U_q \Lambda f = U_q^\dagger \Lambda f + U_q \Lambda f(b) \varphi_q^X = \Lambda \left(V_q^\dagger f + V_q f(b) \varphi_q^Y \right) = \Lambda V_q f, \tag{3.22}$$

which proves that $P_t \Lambda = \Lambda Q_t$ on $\mathcal{D}_\Lambda$ and this completes the proof.

Proof of (2) $\Rightarrow$ (3)

Now let us further assume condition (3.9) for Λ and $\widehat{\Lambda}$. We start by recalling from [102, Theorem 1] that for any $f \in \mathrm{L}^2(\mathfrak{m}) \cup \{\mathbf{1}_E\}$,

$$U_q f(b) = \frac{\mathbf{U}_q(f) + \gamma_X f(b)}{\delta_X + q\mathbf{U}_q(\mathbf{1}_E) + q\gamma_X}. \tag{3.23}$$

To this end, we will split the proof into three cases, depending on the value of δ_X and γ_X.

Case 1. $\delta_X > 0$. Let us take $f = \mathbf{1}_E$, then under the condition $\Lambda\mathbf{1}_E = \mathbf{1}_E$, we combine (3.6) and (3.17) to get, for any $x \in E$,

$$\begin{aligned} U_q\Lambda\mathbf{1}_E(x) &= U_q\mathbf{1}_E(x) = U_q^{\dagger}\mathbf{1}_E(x) + \varphi_q^X(x)U_q\mathbf{1}_E(b) = \frac{1}{q} - \frac{\varphi_q^X(x)}{q} + \varphi_q^X(x)U_q\mathbf{1}_E(b) \\ &= \frac{1}{q} + \left(U_q\mathbf{1}_E(b) - \frac{1}{q}\right)\varphi_q^X(x). \end{aligned} \tag{3.24}$$

Note that V_q satisfies similar identities as (3.6) and (3.24), hence by linearity of Λ, we have

$$\Lambda V_q\mathbf{1}_E(x) = \frac{1}{q} + \left(V_q\mathbf{1}_E(b) - \frac{1}{q}\right)\Lambda\varphi_q^Y(x).$$

Since $U_q\Lambda f = \Lambda V_q f$ by Lemma 3.3.1, we have

$$\left(U_q\mathbf{1}_E(b) - \frac{1}{q}\right)\varphi_q^X(x) = \left(V_q\mathbf{1}_E(b) - \frac{1}{q}\right)\Lambda\varphi_q^Y(x). \tag{3.25}$$

Moreover, by taking $f = \mathbf{1}_E$ in (3.23), we see that, under the assumption $\delta_X > 0$,

$$U_q\mathbf{1}_E(b) - \frac{1}{q} = \frac{\mathbf{U}_q(\mathbf{1}_E) + \gamma_X}{\delta_X + q\mathbf{U}_q(\mathbf{1}_E) + q\gamma_X} - \frac{1}{q} = -\frac{q^{-1}\delta_X}{\delta_X + q\mathbf{U}_q(\mathbf{1}_E) + q\gamma_X} < 0.$$

On the other hand, using the intertwining relation (2) and the assumptions that $\Lambda Q_t f(b) = Q_t f(b), \Lambda\mathbf{1}_E \equiv \mathbf{1}_E$, we have

$$U_q\mathbf{1}_E(b) = U_q\Lambda\mathbf{1}_E(b) = \Lambda V_q\mathbf{1}_E(b) = V_q\mathbf{1}_E(b),$$

which is a strictly less than $\frac{1}{q}$ if $\delta_X > 0$. Therefore we can easily conclude from (3.25) that $\varphi_q^X(x) = \Lambda\varphi_q^Y(x)$. The dual argument $\varphi_q^{\widehat{Y}}(x) = \widehat{\Lambda}\varphi_q^{\widehat{X}}(x)$ on $E_b\backslash S$ is proved similarly using the dual intertwining relation $\widehat{\Lambda}\widehat{P}_t = \widehat{Q}_t\widehat{\Lambda}$, which can be shown via similar methods as (3.19), and the Markov property equation (3.7) for $\widehat{U}_q$ and $\widehat{V}_q$.

Case 2. $\delta_X = 0, \gamma_X > 0$. Since b is regular, we have that $U_q^{\dagger}\mathbf{1}_{\{b\}}(x) = 0$ for any $x \in E$, and therefore

$$U_q\mathbf{1}_{\{b\}}(x) = \varphi_q^X(x)U_q\mathbf{1}_{\{b\}}(b). \tag{3.26}$$

Recalling the condition $\Lambda\mathbf{1}_{\{b\}} \equiv \mathbf{1}_{\{b\}}$, we therefore have

$$\varphi_q^X(x)U_q\mathbf{1}_{\{b\}}(b) = U_q\mathbf{1}_{\{b\}}(x) = U_q\Lambda\mathbf{1}_{\{b\}}(x) = \Lambda V_q\mathbf{1}_{\{b\}}(x) = V_q\mathbf{1}_{\{b\}}(b)\Lambda\varphi_q^Y(x),$$

where for the last identity we used a similar argument as in (3.26) for V_q. Moreover, taking $f = \mathbf{1}_{\{b\}}$ in (3.23) with $\delta_X = 0$, we have

$$U_q\mathbf{1}_{\{b\}}(b) = \frac{\mathbf{U}_q(\mathbf{1}_{\{b\}}) + \gamma_X\mathbf{1}_{\{b\}}(b)}{q\mathbf{U}_q(\mathbf{1}_E) + q\gamma_X} = \frac{\gamma_X}{q\mathbf{U}_q(\mathbf{1}_E) + q\gamma_X} > 0.$$

Moreover, the assumption $\Lambda Q_t(b) = Q_tf(b)$ yields that

$$U_q\mathbf{1}_{\{b\}}(b) = U_q\Lambda\mathbf{1}_{\{b\}}(b) = \Lambda V_q\mathbf{1}_{\{b\}}(b) = V_q\mathbf{1}_{\{b\}}(b) > 0,$$

therefore $\varphi_q^X(x) = \Lambda\varphi_q^Y(x)$. We can prove $\varphi_q^{\widehat{Y}}(x) = \widehat{\Lambda}\varphi_q^{\widehat{X}}(x)$ on $E_b\backslash S$ using similar techniques with the dual intertwining relation $\widehat{\Lambda}\widehat{P}_t = \widehat{Q}_t\widehat{\Lambda}$ and identity (3.7).

Case 3. $\delta_X = \gamma_X = 0$. Recall that $(\mathbf{P}_t)_{t>0}$ is the (Maisonneuve) entrance law of $P^{\dagger}$, and define $\tilde{\mathbf{Q}}_t$ be such that $\tilde{\mathbf{Q}}_t(f) = \mathbf{P}_t(\Lambda f)$. Our aim is to show that $\tilde{\mathbf{Q}}_t$ is indeed the Maisonneuve entrance law of $Q^{\dagger}$. To this end, we define the measure $\tilde{\mathbf{V}}_0$ on E_b be such that

$$\tilde{\mathbf{V}}_0(f) = \int_0^{\infty} \tilde{\mathbf{Q}}_s(f)ds.$$

Note that $\tilde{\mathbf{V}}_0(f) = \mathbf{U}_0(\Lambda f)$ as by definition, $\mathbf{U}_0(f) = \int_0^{\infty} \mathbf{P}_s(f)ds$. Using the fact that $Q^{\dagger}$ is the minimal semigroup, i.e. $Q_t^{\dagger}f \le Q_tf$ for $f \ge 0$, and together with the intertwining relation (2), we have for all $f \ge 0$,

$$\tilde{\mathbf{V}}_0(Q_t^{\dagger}f) \le \tilde{\mathbf{V}}_0(Q_tf) = \mathbf{U}_0(\Lambda Q_tf) = \mathbf{U}_0(P_t\Lambda f). \tag{3.27}$$

By [51, Corollary 3.23], we can write $\mathbf{U}_0 = \varphi^{\widehat{X}}\mathfrak{m}|_{E_b}$. Moreover, it is well-known that $\varphi^{\widehat{X}}$ is an excessive function of $\widehat{P}$, hence for any $f \in \mathrm{L}^2(\mathfrak{m})$,

$$\varphi^{\widehat{X}}\mathfrak{m}P_tf = (\varphi^{\widehat{X}}, P_tf)_{\mathfrak{m}} = (\widehat{P}_t\varphi^{\widehat{X}}, f)_{\mathfrak{m}} \le (\varphi^{\widehat{X}}, f)_{\mathfrak{m}}.$$

In other words, the measure $\varphi^{\widehat{X}}\mathfrak{m}$ is an excessive measure for P. However, since we are under the case $\gamma_X = 0$, which means $\{b\}$ is a null set for $\mathfrak{m}$, we see from (3.27) that, for $f \geq 0$,

$$\tilde{\mathbf{V}}_0(Q_t^\dagger f) \leq \mathbf{U}_0(P_t\Lambda f) = \varphi^{\widehat{X}}\mathfrak{m}P_t\Lambda f \leq \varphi^{\widehat{X}}\mathfrak{m}\Lambda f = \mathbf{U}_0(\Lambda f) = \tilde{\mathbf{V}}_0(f).$$

Moreover, $\tilde{\mathbf{V}}_0(Q_t^\dagger f) \to 0$ as $t \to \infty$, so $\tilde{\mathbf{V}}_0$ is a purely excessive measure for $Q^\dagger$. Hence by a standard result, see e.g. [53, Theorem 5.25], $\tilde{\mathbf{V}}_0$ is the integral of a uniquely determined entrance law, therefore $\tilde{\mathbf{Q}}_t$ is an entrance law of $Q^\dagger$. Furthermore, let $\tilde{\mathbf{V}}_q = \int_0^\infty e^{-qt}\tilde{\mathbf{Q}}_t dt$, then by [102], the decomposition of resolvents yields

$$V_q f(b) = \Lambda V_q f(b) = U_q\Lambda f(b) = \frac{\mathbf{U}_q(\Lambda f)}{q\mathbf{U}_q(\mathbf{1}_{E\setminus\{b\}})} = \frac{\tilde{\mathbf{V}}_q(f)}{q\tilde{\mathbf{V}}_q(\mathbf{1}_{E\setminus\{b\}})},$$

where we used the fact that $\Lambda\mathbf{1}_{E\setminus\{b\}} = \Lambda(\mathbf{1}_E - \mathbf{1}_{\{b\}}) = \mathbf{1} - \mathbf{1}_{\{b\}} = \mathbf{1}_{E\setminus\{b\}}$. Hence $\tilde{\mathbf{Q}}_t$ is indeed the Maisonneuve entrance law of $Q^\dagger$ and $\mathbf{V}_q \equiv \tilde{\mathbf{V}}_q$. Finally, we use the relation $\mathbf{V}_q = \varphi_q^{\widehat{Y}} m|_{E_b}$, see [51, (3.22)], to get that for any $q > 0, f \in \mathrm{L}^2(m) \cap \mathrm{B}_b(\mathbb{R}_+)^+(E)$,

$$\left\langle \varphi_q^{\widehat{Y}}, f \right\rangle_m = \mathbf{V}_q(f) = \mathbf{U}_q(\Lambda f) = \left\langle \varphi_q^{\widehat{X}}, \Lambda f \right\rangle_{\mathfrak{m}} = \left\langle \widehat{\Lambda}\varphi_q^{\widehat{X}}, f \right\rangle_m,$$

which yields $\varphi_q^{\widehat{Y}}(x) = \widehat{\Lambda}\varphi_q^{\widehat{X}}(x)$ m-a.e. for all $q > 0$. The dual relation works similarly.

Proof of (2) $\Rightarrow$ (1)

Since (2) implies that $U_q\Lambda f = \Lambda V_q f$, and we further have $U_q\Lambda f(b) = \Lambda V_q f(b) = V_q f(b)$ under the assumption $\Lambda Q_t f(b) = Q_t f(b)$ for all $f \in \mathcal{D}_\Lambda$, hence by simply reordering the strong Markov identity (3.6), we have

$$U_q^\dagger \Lambda f(x) = U_q\Lambda f(x) - U_q\Lambda f(b)\varphi_q^X(x) = \Lambda\left(V_q f(x) - V_q f(b)\varphi_q^Y(x)\right) = \Lambda V_q^\dagger f(x),$$

where the second identity uses the fact that (2) $\Rightarrow$ (3). This proves the desired argument.

3.3.2 Proof of corollaries

Proof of Corollary 3.2.1. First, by Theorem 3.2.1, we have $\Phi_X(q) = \Phi_Y(q)$ and therefore,

$$\gamma_Y = \lim_{q\to\infty} \frac{\Phi_Y(q)}{q} = \lim_{q\to\infty} \frac{\Phi_X(q)}{q} = \gamma_X.$$

Moreover, recall that for all $f \in \mathrm{L}^2(\mathfrak{m}) \cup \{\mathbf{1}_E\}$, $U_q f(b)$ can be expressed as (3.23), where γ_X represents the stickiness of X at point b, and similar expression holds for $V_q f(b)$. In other words, when $\gamma_X = \gamma_Y = 0$, b is a reflecting boundary for both X and Y, hence both processes have a Neumann boundary condition at b. While when $\gamma_X = \gamma_Y > 0$, both X and Y have a Robin boundary condition at b and this completes the proof. $\square$

Remark 3.3.1. *If Λ is a bounded operator with $\mathcal{D}_\Lambda = \mathrm{L}^2(m)$, we can also prove this result via infinitesimal generators. In particular, let $\mathbf{L}$ (resp. $\mathbf{G}$) denote the infinitesimal generator of P (resp. Q) in $\mathrm{L}^2(\mathfrak{m})$ (resp. $\mathrm{L}^2(m)$), and $\mathcal{D}(\mathbf{L})$ (resp. $\mathcal{D}(\mathbf{G})$) for its domain. Then for any $f \in \mathcal{D}(\mathbf{G})$, by the definition of infinitesimal generators, we have $\lim_{t\to 0} \frac{Q_t f - f}{t} = \mathbf{G} f$ in $\mathrm{L}^2(m)$. On the other hand, since $\Lambda \in \mathbf{B}(\mathrm{L}^2(m), \mathrm{L}^2(\mathfrak{m}))$, we see that for any sequence $t_n \to 0$ and $n, k \in \mathbb{N}$,*

$$\left\| \Lambda \frac{Q_{t_n} f - f}{t_n} - \Lambda \frac{Q_{t_k} f - f}{t_k} \right\|_{\mathfrak{m}} \leq |||\Lambda||| \left\| \frac{Q_{t_n} f - f}{t_n} - \frac{Q_{t_k} f - f}{t_k} \right\|_m \to 0,$$

which implies that $\left(\Lambda \frac{Q_{t_n} f - f}{t_n}\right)_{n\geq 0}$ is a Cauchy sequence in $\mathrm{L}^2(\mathfrak{m})$, and hence convergent. Since Λ is also a closed operator, we have

$$\Lambda \mathbf{G} f = \Lambda \lim_{t\to 0} \frac{Q_t f - f}{t} = \lim_{t\to 0} \frac{\Lambda Q_t f - \Lambda f}{t} = \lim_{t\to 0} \frac{P_t \Lambda f - \Lambda f}{t}, \tag{3.28}$$

where the last identity comes from assumption (2)*. Moreover, since* Λ *maps* $\mathrm{L}^2(m)$ *to* $\mathrm{L}^2(\mathfrak{m})$*, we have* $\Lambda \mathbf{G} f \in \mathrm{L}^2(\mathfrak{m})$ *and therefore the right-hand side of the above equation converges in* $\mathrm{L}^2(\mathfrak{m})$*. Hence we conclude that* $\Lambda f \in \mathcal{D}(\mathbf{L})$ *and* $\mathbf{L}\Lambda f = \Lambda \mathbf{G} f$ *on* $\mathcal{D}(\mathbf{G})$*. As both* $\mathbf{L}$ *and* $\mathbf{G}$ *have Robin boundary condition at* b *when* $\gamma_X = \gamma_Y > 0$*, this completes the proof.*

Proof of Corollary 3.2.2. First, we combine the representation of Φ_X as in (3.4) and the statement in Theorem 3.2.1 to make the easy observation that

$$\mu_X(dy) = \mu_Y(dy). \tag{3.29}$$

Hence by [51, Corollary 2.22], we have

$$\mathbf{P}(T_b^X \in A) = \mu_X(A) = \mu_Y(A) = \mathbf{Q}(T_b^Y \in A).$$

Note that although the normalizing constants $c(\mathfrak{m})$ and $c(m)$ are not 1 in [51], this will not bring any issue because the Maisonneuve excursion measure $\mathbf{P}$ and $\mathbf{Q}$ are defined up to a multiplicative constant, i.e. if $(\mathbf{P}, \mathfrak{l}^X)$ is an exit system, then so is $(c^{-1}\mathbf{P}, c\mathfrak{l}^X)$ for any $c > 0$. To see this in more detail, we can simply replace $\mathfrak{l}^X$ by $c(\mathfrak{m})\mathfrak{l}^X$ and $\mathbf{P}$ by $\mathbf{P}/c(\mathfrak{m})$, and note that μ_X is also replaced by $\mu_X/c(\mathfrak{m})$. Similar arguments hold for process $\mathfrak{l}^Y$ and $\mathbf{Q}$ as well, which proves the first item. Moreover, denoting $\overline{\mu}_X(c) = \mu_X(c, \infty)$ for any $c > 0$, it is easy to see from (3.29) that $\overline{\mu}_X(c) = \overline{\mu}_Y(c)$ for any $c > 0$. Therefore, by Bertoin [14, Section IV.2 Lemma 1], for any $b \geq a$, we have

$$\mathbb{P}(l_X(a) > b) = \frac{\overline{\mu}_X(b)}{\overline{\mu}_X(a)} = \frac{\overline{\mu}_Y(b)}{\overline{\mu}_Y(a)} = \mathbb{P}(l_Y(a) > b), \tag{3.30}$$

which proves the second item. Finally, for the last item, we simply apply [51, Proposition 2.17] to get, for any $x, q > 0$, that

$$\mathbb{E}_x[e^{-q\zeta_X}] = \frac{\delta_X}{\Phi_X(q)} = \frac{\delta_Y}{\Phi_Y(q)} = \mathbb{E}_x[e^{-q\zeta_Y}],$$

Hence ζ_X and ζ_Y have the same distribution under $\mathbb{P}_x$ and this concludes the proof of this Proposition. $\square$

Proof of Corollary 3.2.3. Given the intertwining relation in (1), by Theorem 3.2.1, we see that $\Phi_X = \Phi_Y$. Moreover, assuming that Y is a quasi-diffusion, which means that μ_Y has an absolutely continuous density u_Y which admits the representation (3.10) for some measure ν_Y, hence so does μ_X since we can simply take $\nu_X = \nu_Y$. On the other hand, since Y has the Krein's property, $Q_t^\dagger$ satisfies the expansion given in (3.12), and there exist functions $(h_q)_{q\in\sigma(\mathbf{G}^\dagger)}$ such that

$$\left\langle d\mathtt{E}_q^Y f, g\right\rangle_m = (f, h_q)_m (g, h_q)_m \nu_Y(dq),$$

for any $f, g \in \mathrm{L}^2(m)$. Now let us define the family of operators $(\mathtt{E}_q^X)_{q\in\sigma(\mathbf{G}^\dagger)}$ as $\mathtt{E}_q^X = \Lambda \mathtt{E}_q^Y \Lambda^{-1}$ on $\mathcal{D}(\mathtt{E}^X) = \mathrm{Ran}(\Lambda)$. For any $f \in \mathcal{D}(\mathtt{E}^X)$, let $g = \Lambda^{-1} f \in \mathcal{D}_\Lambda$, and we observe the following.

(i) $\mathcal{D}(\mathtt{E}^X) = \mathrm{Ran}(\Lambda)$ is assumed to be dense in $\mathrm{L}^2(\mathfrak{m})$. Moreover, for any $q \in \sigma(\mathbf{G}^\dagger)$, we have $\mathtt{E}_q^Y g \in \mathcal{D}_\Lambda$ by assumption. Hence

$$\mathtt{E}_q^X f = \Lambda \mathtt{E}_q^Y \Lambda^{-1} f = \Lambda \mathtt{E}_q^Y g \in \mathcal{D}(\mathtt{E}^X),$$

i.e. $\mathtt{E}_q^X \mathcal{D}(\mathtt{E}^X) \subseteq \mathcal{D}(\mathtt{E}^X)$.

(ii) Using the property of the resolution of identity $\mathtt{E}^Y$ and the boundedness of Λ, we have

$$\begin{aligned}
\lim_{q\to\inf\sigma(\mathbf{G}^\dagger)} \mathtt{E}_q^X f &= \lim_{q\to\inf\sigma(\mathbf{G}^\dagger)} \Lambda \mathtt{E}_q^Y \Lambda^{-1} f = \lim_{q\to\inf\sigma(\mathbf{G}^\dagger)} \Lambda \mathtt{E}_q^Y g = 0, \\
\lim_{q\to\sup\sigma(\mathbf{G}^\dagger)} \mathtt{E}_q^X f &= \lim_{q\to\sup\sigma(\mathbf{G}^\dagger)} \Lambda \mathtt{E}_q^Y \Lambda^{-1} f = \Lambda\Lambda^{-1} f = f.
\end{aligned}$$

(iii) $\mathtt{E}_q^X \mathtt{E}_r^X f = \Lambda \mathtt{E}_q^Y \Lambda^{-1} \Lambda \mathtt{E}_r^Y \Lambda^{-1} f = \Lambda \mathtt{E}_{\min(q,r)}^Y \Lambda^{-1} f = \mathtt{E}_{\min(q,r)}^X f$ for any $q, r \in \sigma(\mathbf{G}^\dagger)$.

Hence $\mathtt{E}^X$ is a non-self-adjoint resolution of identity. Next, let $(q_k)_{k=0}^n$ be a partition of $[\inf \sigma(\mathbf{G}^\dagger), \sup \sigma(\mathbf{G}^\dagger)]$. Then for any $f \in \mathcal{D}(\mathtt{E}^X), g \in \mathrm{L}^2(\mathfrak{m})$, since $\mathtt{E}^Y(\Delta_k) = \mathtt{E}^Y_{q_k} - \mathtt{E}^Y_{q_{k-1}}$ is an orthogonal projection, we have

$$\begin{aligned}\sum_{k=1}^n \left|\left\langle [\mathtt{E}^X_{q_k} - \mathtt{E}^X_{q_{k-1}}]f, g\right\rangle\right| &= \sum_{k=1}^n \left|\left\langle \mathtt{E}^Y(\Delta_k)\Lambda^{-1}f, \widehat{\Lambda}g\right\rangle\right| \leq \|\widehat{\Lambda}g\| \sum_{k=1}^n \|\mathtt{E}^Y(\Delta_k)\Lambda^{-1}f\| \\ &\leq \|\widehat{\Lambda}g\| \left(\sum_{k=1}^n \|\mathtt{E}^Y(\Delta_k)\Lambda^{-1}f\|^2\right)^{\frac{1}{2}} = \|\widehat{\Lambda}g\| \left(\sum_{k=1}^n \left\langle \mathtt{E}^Y(\Delta_k)\Lambda^{-1}f, \Lambda^{-1}f\right\rangle\right)^{\frac{1}{2}} \\ &= \|\widehat{\Lambda}g\|\|\Lambda^{-1}f\| \leq \|\widehat{\Lambda}g\|\|\|\Lambda^{-1}\|\|\|f\|\end{aligned}$$

since the series $\sum_{k=1}^n \left\langle \mathtt{E}^Y(\Delta_k)\Lambda^{-1}f, \Lambda^{-1}f\right\rangle$ is telescoping. Therefore, we see that $\left\langle \mathtt{E}^X_{\cdot} f, g\right\rangle$ is of bounded variation on $[\inf \sigma(\mathbf{G}^\dagger), \sup \sigma(\mathbf{G}^\dagger)]$, and by Riesz representation theorem, there exists a unique operator $\tilde{P}^\dagger_t f = \int_{\sigma(\mathbf{G}^\dagger)} e^{-qt} d\mathtt{E}^X_q f$ on $\mathcal{D}(\mathtt{E}^X)$. Then it is easy to see that for $f \in \mathcal{D}(\mathtt{E}^X)$, $g \in \mathrm{L}^2(\mathfrak{m})$,

$$\begin{aligned}\left\langle \tilde{P}^\dagger_t f, g\right\rangle_{\mathfrak{m}} &= \int_0^\infty e^{-qt} d\left\langle \mathtt{E}^X_q f, g\right\rangle_{\mathfrak{m}} = \int_0^\infty e^{-qt} d\left\langle \Lambda \mathtt{E}^Y_q \Lambda^{-1} f, g\right\rangle_{\mathfrak{m}} = \int_0^\infty e^{-qt} d\left\langle \mathtt{E}^Y_q \Lambda^{-1} f, \widehat{\Lambda}g\right\rangle_m \\ &= \left\langle Q^\dagger_t \Lambda^{-1} f, \widehat{\Lambda}g\right\rangle_m = \left\langle P^\dagger_t \Lambda\Lambda^{-1} f, g\right\rangle_{\mathfrak{m}} = \left\langle P^\dagger_t f, g\right\rangle_{\mathfrak{m}},\end{aligned}$$

which shows that indeed $P^\dagger_t f = \tilde{P}^\dagger_t f$ on $\mathcal{D}(\mathtt{E}^X)$. Moreover, for any $f \in \mathcal{D}(\mathtt{E}^X), g \in \mathrm{L}^2(\mathfrak{m})$,

$$\begin{aligned}\left\langle d\mathtt{E}^X_q f, g\right\rangle_{\mathfrak{m}} &= \left\langle \Lambda d\mathtt{E}^Y_q \Lambda^{-1} f, g\right\rangle_{\mathfrak{m}} = \left\langle d\mathtt{E}^Y_q \Lambda^{-1} f, \widehat{\Lambda}g\right\rangle_m \\ &= (\Lambda^{-1}f, h_q)_m (\Lambda g, h_q)_m \nu_Y(dq) = (\Lambda^{-1}f, h_q)_m (\Lambda g, h_q)_m \nu_X(dq),\end{aligned}$$

which means that $\left\langle d\mathtt{E}^X_q f, g\right\rangle_{\mathfrak{m}}$ is absolutely continuous with respect to ν_X and this shows that X (or its semigroup P) also satisfies the weak-Krein property. □

3.4 Reflected self-similar and generalized Laguerre semigroups

The aim of this part is two-fold. On the one hand, we illustrate the main results of the previous sections by studying two important classes of Markov processes, namely the spectrally negative positive self-similar Markov processes that were introduced by Lamperti [69] and their associated generalized Laguerre processes whose definition will be recalled below. We emphasize that these two classes have been studied intensively over the last two decades and appear in many recent studies in applied mathematics, such as random planar maps, fragmentation equation, biology, see e.g. [15], [16] and [91]. On the other hand, we also provide the spectral expansion of both the minimal and reflected semigroups associated to the generalized Laguerre processes. This complements the work of Patie and Savov [91] where such analysis has been carried out for the transient with infinite lifetime generalized Laguerre semigroups. From now on, we fix the Lusin space to be $(E, \mathcal{E}) = (\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$, the space of Borel sets on non-negative real numbers, and we set $b = 0$. Next, we denote by $\overline{Y} = (\overline{Y}_t)_{t\geq 0}$ the squared Bessel process with parameter $-\theta$, with $\theta \in (0,1)$, and write $\overline{Q} = (\overline{Q}_t)_{t\geq 0}$ its corresponding semigroup, i.e. $\overline{Q}_t f(x) = \mathbb{E}_x[f(\overline{Y}_t)]$, $f \in \mathrm{C}_0(\mathbb{R}_+)$, $x, t \geq 0$. It is well known, see e.g. [23, Chapter IV.6], that $\overline{Q}$ is a Feller semigroup, whose infinitesimal generator is given by

$$\overline{\mathbf{G}} f(x) = x f^{''}(x) + (1-\theta) f^{'}(x), \quad x > 0,$$

for $f \in \mathcal{D}(\overline{\mathbf{G}}) = \{f \in \mathrm{C}_0(\mathbb{R}_+); \overline{\mathbf{G}} f \in \mathrm{C}_0(\mathbb{R}_+), f^+(0) = 0\}$ where $f^+(x) = \lim_{h\downarrow 0} \frac{f(x+h)-f(x)}{s(x+h)-s(x)}$ is the right derivative of f with respect to the scale function $s(x) = \int^x y^{\theta-1} e^y dy$. Note that $\overline{Q}$ possesses the so-called 1-self-similarity property, i.e. for all $t, x, c > 0$,

$$\overline{Q}_t f(cx) = \overline{Q}_{c^{-1}t} \mathrm{d}_c f(x),$$

where $\mathrm{d}_c f(x) = f(cx)$. Moreover, the measure $\overline{m}(x)dx = x^{-\theta}dx$, $x > 0$, is the unique excessive measure for $\overline{Q}$, and therefore $\overline{Q}$ admits a unique strongly continuous contraction extension on $\mathrm{L}^2(\overline{m})$, also denoted by $\overline{Q}$ when there is no confusion. Furthermore, note that 0 is a regular reflecting boundary for $\overline{Y}$, hence we let

$\overline{Q}^{\dagger} = (\overline{Q}_t^{\dagger})_{t\geq 0}$ denote the $\mathrm{L}^2(\overline{m})$-semigroup of the killed process $(\overline{Y}, T_0^{\overline{Y}})$ where $T_0^{\overline{Y}} = \inf\{t \geq 0; \overline{Y}_t = 0\}$. Now let the process $Y = (Y_t)_{t\geq 0}$ be defined as

$$Y_t = e^{-t}\overline{Y}_{e^t-1}, \quad t \geq 0, \tag{3.31}$$

which is the (classical) Laguerre process of parameter $-\theta$, also known as the squared radial Ornstein-Uhlenbeck process with parameter $-\theta$. Its semigroup $Q = (Q_t)_{t\geq 0}$, which admits the representation

$$Q_t f = \overline{Q}_{e^t-1}\mathrm{d}_{e^{-t}} \circ f, \tag{3.32}$$

is also a Feller semigroup in $\mathrm{C}_0(\mathbb{R}_+)$ with infinitesimal generator given by

$$\mathbf{G}f(x) = xf''(x) + (1-\theta-x)f'(x), \quad x > 0,$$

with $\mathcal{D}(\mathbf{G}) = \{f \in \mathrm{C}_0(\mathbb{R}_+); \mathbf{G}f \in \mathrm{C}_0(\mathbb{R}_+), f^+(0) = 0\}$. Moreover, Q admits an invariant measure $m(x)dx$ with density given by

$$m(x) = \frac{x^{-\theta}e^{-x}}{\Gamma(1-\theta)}, \quad x > 0, \tag{3.33}$$

which is the probability density of a Gamma random variable of parameter $1-\theta$, denoted by $G(1-\theta)$. Therefore, Q admits a strongly continuous contraction extension on $\mathrm{L}^2(m)$, also denoted by Q when there is no confusion. It is well-known that Q is self-adjoint in $\mathrm{L}^2(m)$ with a spectral decomposition given in terms of the (classical) Laguerre polynomials, see e.g. [7, Section 2.7.3]. We also let $Q^{\dagger} = (Q_t^{\dagger})_{t\geq 0}$ be the $\mathrm{L}^2(m)$-semigroup of the killed process (Y, T_0^Y) since 0 is also a reflecting boundary for Y.

We proceed by introducing two classes of Markov processes with jumps which are natural generalizations of the processes $\overline{Y}$ and Y in the sense that they share the 1-self-similarity property of $\overline{Y}$ and the second class is constructed from the first one by means of the relation (3.31). To this end, let $\xi = (\xi_t)_{t\geq 0}$ be a spectrally negative Lévy process, which is possibly killed at a rate $\kappa \geq 0$, that is, killed at an independent exponential time with parameter κ. It is then well-known that ξ can be characterized by its Laplace exponent $\psi : \mathbb{C}_+ = \{z \in \mathbb{C} : \Re(z) \geq 0\} \to \mathbb{C}$, which is defined, for any $\Re(z) \geq 0$, by

$$\psi(z) = \beta z + \frac{\sigma^2}{2}z^2 - \int_0^\infty (e^{-zy} - 1 + zy\mathbf{1}_{|y|<1})\Pi(dy) - \kappa, \tag{3.34}$$

where $\beta \in \mathbb{R}, \sigma \geq 0, \kappa \geq 0$, and Π is a σ-finite measure satisfying $\int_0^\infty (y^2 \wedge y)\Pi(dy) < \infty$. Note that the quadruplet $(\beta, \sigma, \Pi, \kappa)$ uniquely determines ψ and therefore uniquely determines ξ. Furthermore, let

$$\mathbf{T}(t) = \inf\left\{s > 0; \int_0^s e^{\xi_r} dr > t\right\}, \tag{3.35}$$

and for an arbitrary $x > 0$, define the process $\overline{\mathbf{X}} = (\overline{\mathbf{X}}_t)_{t\geq 0}$ by

$$\overline{\mathbf{X}}_t = xe^{\xi_{\mathbf{T}(tx^{-1})}}, \quad t \geq 0, \tag{3.36}$$

where the above quantity is assumed to be 0 when $\mathbf{T}(tx^{-1}) = \infty$. According to Lamperti [69], $\overline{\mathbf{X}}$ is a 1-self-similar Markov process, and its infinitesimal generator takes the form

$$\overline{\mathbf{L}}f(x) = \sigma^2 xf''(x) + \left(\beta + \sigma^2\right) f'(x) + \int_0^\infty \left(f(e^{-y}x) - f(x) + yxf'(x)\right) \frac{\Pi(dy)}{x} - \kappa f(x), \tag{3.37}$$

for at least functions $f \in \mathcal{D}_{\mathbf{L}} = \{f_e(\cdot) = f(e^{\cdot}) = \mathrm{C}^2([-\infty, \infty])\}$. Next, writing the set $\mathcal{N} = \{\psi \text{ of the form (3.34)}\}$, the Lamperti transformation (3.36) enables to define a bijection between the subspace of negative definite functions $\mathcal{N}$ and the 1-self-similar processes $\overline{\mathbf{X}}$. Moreover, when

$$\psi \in \mathcal{N}_\uparrow \quad = \quad \{\psi \in \mathcal{N};\ \beta \geq 0, \kappa = 0\} \tag{3.38}$$

then $\overline{\mathbf{X}}$ never reaches 0 and has an a.s. infinite lifetime. Otherwise, if $\psi \in \mathcal{N} \setminus \mathcal{N}_\uparrow$, then 0 is an absorbing point, which is reached continuously if $\kappa = 0$ and $\beta < 0$ or by a jump if $\kappa > 0$. In addition, according to Rivero [101], see also Fitzsimmons [50], for each

$$\psi \in \mathcal{N}_{\checkmark} = \left\{\psi \in \mathcal{N};\ \exists\, \theta \in (0,1) \text{ such that } \psi(\theta) = 0 \text{ and } \int_{x>1} x e^{\theta x} \Pi(dx) < \infty\right\},$$

$\overline{\mathbf{X}}$ admits a unique recurrent extension that leaves a.s. 0 continuously, denoted by $\overline{X} = (\overline{X}_t)_{t\geq 0}$. Its minimal process $\overline{X}^{\dagger} = (\overline{X}^{\dagger}_t)_{t\geq 0} = (\overline{X}_t; 0 \leq t \leq T_0^{\overline{X}})$ is equivalent to $\overline{\mathbf{X}}$, and 0 is a regular boundary for $\overline{X}$. Let $\overline{P} = (\overline{P}_t)_{t\geq 0}$ and $\overline{P}^{\dagger} = (\overline{P}^{\dagger}_t)_{t\geq 0}$ denote the Feller semigroups of $\overline{X}$ and $\overline{X}^{\dagger}$, respectively, i.e. $\overline{P}_t f(x) = \mathbb{E}_x[f(\overline{X}_t)], \overline{P}^{\dagger}_t f(x) = \mathbb{E}_x[f(\overline{X}_t), t < T_0^{\overline{X}}], f \in \mathrm{C}_0(\mathbb{R}_+)$. We also deduce from [101, Lemma 3] that $\overline{m}$ is, up to a multiplicative constant, the unique excessive measure for $\overline{P}$ and also an excessive measure for $\overline{P}^{\dagger}$, hence both $\overline{P}$ and $\overline{P}^{\dagger}$ can be uniquely extended to a strongly continuous contraction semigroup on $\mathrm{L}^2(\overline{m})$, still using the same notations when there is no confusion.

Moreover, we define the process $X = (X_t)_{t\geq 0}$ by $X_t = e^{-t}\overline{X}_{e^t-1}, t \geq 0$, which, by the self-similarity property of $\overline{X}$ is a homogeneous Markov process and is called a *reflected generalized Laguerre process*, with 0 also being a regular boundary. $X^{\dagger} = (X^{\dagger}_t)_{t\geq 0}$ stands for its minimal process, that is the one killed at the stopping time T_0^X. Note that, due to the deterministic and bijective transform between processes X and $\overline{X}$, X can also be uniquely characterized by $\psi \in \mathcal{N}_{\checkmark}$. We further let $P = (P_t)_{t\geq 0}$ and $P^{\dagger} = (P^{\dagger}_t)_{t\geq 0}$ denote the Feller semigroups of X and $X^{\dagger}$, respectively. Then we easily get that

$$P_t f = \overline{P}_{e^t-1} \mathrm{d}_{e^{-t}} \circ f, \tag{3.39}$$

and the infinitesimal generator of P is given, for $f \in \mathcal{D}_{\mathbf{L}}$, by

$$\mathbf{L} f(x) = \overline{\mathbf{L}} f(x) - x f'(x). \tag{3.40}$$

We observe that $\overline{Y}$ and Y are special instances of $\overline{X}$ and X respectively, when $\kappa = 0$ and $\Pi \equiv 0$ in (3.37). Before stating the main result of this section, we need to introduce a few additional objects. First, we recall that the Wiener-Hopf factorization for spectrally negative Lévy processes, see e.g. [67], yields that the function ϕ defined by $\phi(u) = \frac{\psi(u)}{u-\theta}, u \geq 0$, is a Bernstein function, that is the Laplace exponent of a subordinator $\eta = (\eta_t)_{t\geq 0}$ (i.e. a non-decreasing Lévy process), see e.g. [107] for an excellent account of Bernstein functions. Then, for $f \in \mathrm{C}_0(\mathbb{R}_+)$ we define the Markov multiplier Λ_ϕ by

$$\Lambda_\phi f(x) = \mathbb{E}[f(xI_\phi)] \tag{3.41}$$

where $I_\phi = \int_0^\infty e^{-\eta_t} dt$ is the so-called exponential functional of η, see e.g. [90] and the references therein for a recent account on this variable. We are now ready to state the following.

Theorem 3.4.1. *For each $\psi \in \mathcal{N}_{\checkmark}$, the following statements hold.*

1. *There exists a positive random variable V_Ψ whose law is absolutely continuous with a density denoted by $\mathfrak{m}$, and it is an invariant measure for the semigroup P. Moreover, the law of V_Ψ is determined by its entire moments*

$$\mathcal{M}_{V_\Psi}(n+1) = \prod_{k=1}^{n} \frac{\psi(k)}{k}, \quad n \in \mathbb{N}. \tag{3.42}$$

2. *$\Lambda_\phi \in \mathbf{B}(\mathrm{C}_0(\mathbb{R}_+)) \cap \mathbf{B}(\mathrm{L}^2(\overline{m})) \cap \mathbf{B}(\mathrm{L}^2(m), \mathrm{L}^2(\mathfrak{m}))$ and has a dense range in both $\mathrm{L}^2(\overline{m})$ and $\mathrm{L}^2(\mathfrak{m})$. Furthermore, both Λ_ϕ and $\widehat{\Lambda}_\phi$ are mass-preserving and satisfy the condition (3.9).*

3. *For all $f \in \mathrm{L}^2(\overline{m})$ (resp. $f \in \mathrm{L}^2(m)$), we have*

$$\overline{P}_t \Lambda_\phi f = \Lambda_\phi \overline{Q}_t f \quad (resp.\ P_t \Lambda_\phi f = \Lambda_\phi Q_t f), \tag{3.43}$$

and consequently,

$$\overline{P}_t^\dagger \Lambda_\phi f = \Lambda_\phi \overline{Q}_t^\dagger f \quad (resp.\ P_t^\dagger \Lambda_\phi f = \Lambda_\phi Q_t^\dagger f). \tag{3.44}$$

4. *Under the normalization $c(\overline{m}) = c(\mathfrak{m}) = c(m) = 1$, we have for any $q > 0$,*

$$\Phi_{\overline{Y}}(q) = \Phi_{\overline{X}}(q) = \frac{\Gamma(1-\theta)}{\Gamma(\theta)} 2^{1-\theta} q^{\theta}, \quad \Phi_X(q) = \Phi_Y(q) = \frac{\theta\Gamma(q+\theta)}{\Gamma(1+\theta)\Gamma(q)}. \tag{3.45}$$

5. *$\overline{X}$ and X satisfy the weak-Krein property.*

Remark 3.4.1. *(i) The expression of the entire moment of V_ψ appears in the work of Barczy and Döering [8, Theorem 1]. Their proof rely on a representation as the solution of stochastic differential equation of some recurrent extensions of Lamperti processes. We shall provide an alternative proof which is in the spirit of the papers of Rivero [101] and Fitzsimmons [50] and could be used in a more general context.*

(ii) To prove (3.43)*, we shall resort to a criteria that was developed in [31], and the details of this proof can be found in Section 3.4.1. Note that a crucial assumption is the conservativeness of the semigroups (i.e. $\overline{P}_t\mathbf{1} = \mathbf{1}, P_t\mathbf{1} = \mathbf{1}$), a property that is not fulfilled by $\overline{P}^{\dagger}$ or $P^{\dagger}$. Instead, to prove* (3.44)*, we use our Theorem 3.2.1, revealing its usefulness in this context.*

(iii) It is well-known that the local time is defined up to a normalization constant. In this paper, it is considered as an additive functional whose support is $\{0\}$ and with the total mass of its asociated Revuz measure normalized to $c(\overline{m}) = c(\mathfrak{m}) = c(m) =$ 1. However, one can also view the local times of $\overline{Y}$ and Y as the unique increasing process in the Doob-Meyer decomposition of the semi-martingale $(\overline{Y}_t^{\theta})_{t\geq 0}$ and $(Y_t^{\theta})_{t\geq 0}$ respectively, see e.g. [58, Theorem 3.2], which are denoted by $\tilde{\mathfrak{l}}^{\overline{Y}}$ and $\tilde{\mathfrak{l}}^{Y}$. The local times for $\overline{X}$ and X can be defined similarly, see Section 3.4.2 for the proof. Under this definition, the total mass of the Revuz measure is given by

$$\tilde{c}(\mathfrak{m}) = \frac{\theta W_\phi(1+\theta)}{\Gamma(1-\theta)\Gamma(1+\theta)}, \quad \tilde{c}(m) = \frac{\theta}{\Gamma(1-\theta)}, \tag{3.46}$$

where W_ϕ will be defined later in the context. Under this normalization, the corresponding Laplace exponents take the form

$$\tilde{\Phi}_X(q) = \frac{\Gamma(1-\theta)\Gamma(q+\theta)}{W_\phi(1+\theta)\Gamma(q)}, \quad \tilde{\Phi}_Y(q) = \frac{\Gamma(1-\theta)}{\Gamma(1+\theta)}\frac{\Gamma(q+\theta)}{\Gamma(q)}. \tag{3.47}$$

We will detail this computation in Section 3.4.2.

(iv) The intertwining relation (3.43) *is also a useful tool for deriving the spectral expansion of $P_t f$ and $P_t^\dagger f$ in* $\mathrm{L}^2(\mathfrak{m})$ *under various conditions. We will provide such expansions in Section 3.4.3.*

The rest of this section is devoted to proving Theorem 3.4.1.

3.4.1 Proof of Theorem 3.4.1(1), (2) and (3)

First, let us prove that the expression of the entire moments of the variable $\overline{X}_1$ under $\mathbb{P}_0$ is given by (3.42). Writing $\psi_\uparrow(u) = \psi(u+\theta), u \geq 0$, we observe that

$$\psi_\uparrow(0) = \psi(\theta) = 0, \quad \psi_\uparrow(u) > 0 \text{ for } u > 0, \quad \psi_\uparrow^{'}(0+) = \psi^{'}(\theta) > 0,$$

hence $\psi_\uparrow \in \mathcal{N}_\uparrow$ is the Laplace exponent of a spectrally negative Lévy process $\xi^\uparrow$, which drifts to $+\infty$ a.s. and is associated, via the Lamperti mapping, to a 1-self-similar process which can be viewed as the minimal process $X^\dagger$ conditioned to stay positive. Let $I_{\psi_\uparrow} = \int_0^\infty e^{-\xi_t^\uparrow} dt$ denote the exponential functional of $\xi^\uparrow$, which, by [18, Theorem 1], is well-defined, i.e. $I_{\psi_\uparrow} < \infty$ a.s., and has negative moments of all orders, see [18, Theorem 3]. We also let $\overline{U}_q f(x) = \int_0^\infty e^{-qt}\overline{P}_t f(x) dt$ denote the resolvent of the self-similar semigroup $\overline{P}$. Then combining [101, Theorem 2] and [18, Equation (4)], with $p_z(x) = x^z$, we get, for each $q > 0, \Re(z) \geq 0$,

$$\begin{aligned}\overline{U}_q p_z(0) &= \frac{1}{\mathcal{M}_{I_{\psi_\uparrow}}(\theta)\Gamma(1-\theta)q^\theta}\mathcal{M}_{I_{\psi_\uparrow}}(-z+\theta)\int_0^\infty e^{-qt}t^{z-\theta}dt \\ &= \frac{\Gamma(z-\theta+1)}{\Gamma(1-\theta)}\frac{\mathcal{M}_{I_{\psi_\uparrow}}(-z+\theta)}{\mathcal{M}_{I_{\psi_\uparrow}}(\theta)}p_{-z-1}(q).\end{aligned} \tag{3.48}$$

On the other hand, from the definition of the resolvent $\overline{U}_q$ and the 1-self-similarity of $\overline{P}$, we have

$$\overline{U}_q p_z(0) = \int_0^\infty e^{-qt}\overline{P}_t p_z(0)dt = \mathcal{M}_{V_\Psi}(z+1)\int_0^\infty e^{-qt}t^z dt = \mathcal{M}_{V_\Psi}(z+1)\Gamma(z+1)p_{-z-1}(q). \tag{3.49}$$

Combining equation (3.48) and (3.49), we deduce that

$$\mathcal{M}_{V_\Psi}(z+1) = \frac{\Gamma(z-\theta+1)}{\Gamma(1-\theta)\Gamma(z+1)}\frac{\mathcal{M}_{I_{\psi_\uparrow}}(-z+\theta)}{\mathcal{M}_{I_{\psi_\uparrow}}(\theta)} = \mathcal{M}_{B(1-\theta,\theta)}(z+1)\frac{\mathcal{M}_{I_{\psi_\uparrow}}(-z+\theta)}{\mathcal{M}_{I_{\psi_\uparrow}}(\theta)}, \tag{3.50}$$

where $B(1-\theta,\theta)$ is a random variable following a Beta distribution with parameters $(1-\theta,\theta)$. By [89, (2.3)], the Mellin transform of $I_{\psi_\uparrow}$ satisfies the functional equation

$$\mathcal{M}_{I_{\psi_\uparrow}}(-z+1) = \frac{z}{\psi_\uparrow(z)}\mathcal{M}_{I_{\psi_\uparrow}}(-z), \tag{3.51}$$

which holds on the domain $\{z \in \mathbb{C} : \psi_\uparrow(\Re(z)) \le 0\}$. Combining (3.51) and (3.50), we get, for $\Re(z) \ge 0$, that

$$\frac{\mathcal{M}_{V_\Psi}(z+1)}{\mathcal{M}_{V_\Psi}(z)} = \frac{\Gamma(z)}{\Gamma(z+1)}\frac{\Gamma(z-\theta+1)}{\Gamma(z-\theta)}\frac{\mathcal{M}_{I_{\psi_\uparrow}}(-z+\theta)}{\mathcal{M}_{I_{\psi_\uparrow}}(-z+\theta+1)} = \frac{z-\theta}{z}\frac{\psi_\uparrow(z-\theta)}{z-\theta} = \frac{\psi_\uparrow(z-\theta)}{z} = \frac{\psi(z)}{z}.$$

Hence (3.42) can be easily observed from the above relation together with the initial condition $\mathcal{M}_{V_\Psi}(1) = 1$. Next, the estimates

$$\left|\frac{\frac{\prod_{k=1}^{n+1}\psi(k)}{((n+1)!)^2}}{\frac{\prod_{k=1}^{n}\psi(k)}{(n!)^2}}\right| = \left|\frac{\psi(n+1)}{(n+1)^2}\right| \to \begin{cases} \frac{\sigma^2}{2} & \text{if } \sigma^2 > 0 \\ 0 & \text{if } \sigma^2 = 0 \end{cases} \qquad \text{as } n \to \infty,$$

yields that the series

$$\mathbb{E}[e^{qV_\Psi}] = \sum_{n=1}^\infty \frac{\mathcal{M}_{V_\Psi}(n+1)}{n!}q^n = \sum_{n=1}^\infty \frac{\prod_{k=1}^n \psi(k)}{(n!)^2}q^n \tag{3.52}$$

converges for $|q| < \frac{2}{\sigma^2}$ when $\sigma^2 > 0$ and converges for $|q| < \infty$ when $\sigma^2 = 0$. Therefore, we get that V_Ψ is moment determinate. This completes the proof of Theorem 3.4.1(1). Now, combining [101, Theorem 1] and [85, Proposition 2.4] combined, we obtain that the law of V_Ψ is absolute continuous and we denote its density $\mathfrak{m}$. Then, we write, for any $t, x > 0$,

$$t\mathfrak{n}_t(tx) = \mathfrak{m}(x),$$

i.e. changing slightly notation here and below $\int_0^\infty f(x)\mathfrak{m}(x)dx = \mathfrak{m}f = \mathfrak{n}_t \mathrm{d}_{1/t} f$. Then, combining (3.42) with the self-similarity property of $\overline{P}$ identifies $(\mathfrak{n}_t(x)dx)_{t\geq 0}$ as a family of entrance laws for $\overline{P}$, that is, for any $t, s > 0$ and $f \in \mathrm{C}_0(\mathbb{R}_+)$, $\mathfrak{n}_t\overline{P}_s f = \mathfrak{n}_{t+s} f$. Next, using successively the relation (3.39), the previous identity with $t = 1$ and $s = e^t - 1$, and the definition of $\mathfrak{n}_t$ above, we get that, for any $t > 0$,

$$\mathfrak{m}P_t f = \mathfrak{m}\overline{P}_{e^t-1}\mathrm{d}_{e^{-t}} \circ f = \mathfrak{n}_{e^t}\mathrm{d}_{e^{-t}} \circ f = \mathfrak{m}f.$$

Hence, $\mathfrak{m}(x)dx$ is an invariant measure for P. Therefore, P can be uniquely extended to a strongly continuous contraction semigroup on $\mathrm{L}^2(\mathfrak{m})$, also denoted by P when there is no confusion.

Next, we proceed by proving Theorem 3.4.1(2). The fact that $\Lambda_\phi \in \mathbf{B}(\mathrm{C}_0(\mathbb{R}_+))$ follows immediately by dominated convergence. For any $f \in \mathrm{L}^2(\overline{m})$, we use the Cauchy-Schwarz inequality and a change of variable to deduce that

$$\|\Lambda_\phi f\|^2_{\overline{m}} \leq \mathbb{E}\left[\int_0^\infty f^2(xI_\phi)\overline{m}(x)dx\right] = \mathcal{M}_{I_\phi}(\theta)\int_0^\infty f^2(x)\overline{m}(x)dx = \mathcal{M}_{I_\phi}(\theta)\|f\|^2_{\overline{m}}.$$

Since $\mathcal{M}_{I_\phi}(\theta) < \infty$ by [91, Proposition 6.8], we get that $\Lambda_\phi \in \mathbf{B}(\mathrm{L}^2(\overline{m}))$. In order to prove that the range of Λ_ϕ is dense in $\mathbf{B}(\mathrm{L}^2(\overline{m}))$, we first define the following function, for $\Re(z) \in \left(\frac{\theta}{2}, \frac{\theta}{2}+1\right)$,

$$\mathcal{M}_g(z) = \frac{W_\phi(-z+\frac{\theta}{2}+1)\Gamma(z-\frac{\theta}{2})}{\Gamma(-z+\frac{\theta}{2}+1)}, \tag{3.53}$$

where W_ϕ is the unique log-concave solution to the functional equation $W_\phi(z+1) = \phi(z)W_\phi(z)$ for $\Re(z) \geq 0$, with initial condition $W_\phi(1) = 1$, see [91, Theorem 5.1] and [90] for a comprehensive study of this equation. Using the Stirling formula, see e.g. [83, (2.1.8)],

$$|\Gamma(z)| = C|e^{-z}||z^z||z|^{-\frac{1}{2}}(1 + o(1)), \quad C > 0, \tag{3.54}$$

which is valid for large $|z|$ and $|\arg(z)| < \pi$, as well as the large asymptotic behavior, along the imaginary line $\frac{1}{2} + ib$, of W_ϕ, see [91, Theorem 5.1(3)], we have

$$\mathcal{M}_g\left(\frac{1}{2} + ib\right) = \mathrm{o}\left(|b|^{-\theta-u}\right) \tag{3.55}$$

as $|b| \to \infty$, for any $u > \frac{1}{2} - \theta$. $\mathcal{M}_g$ being analytical on the strip $\Re(z) \in \left(\frac{\theta}{2}, \frac{\theta}{2} + 1\right)$, it is therefore absolutely integrable and decays to zero uniformly along the lines of this strip. Hence one can apply the Mellin inversion theorem which combines with the Cauchy's Theorem, see e.g. [93, Lemma 3.1] for details of a similar computation, gives that

$$g(x) = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} x^{-z}\mathcal{M}_g(z)dz = \sum_{n=0}^{\infty} \frac{(-1)^n W_\phi(n+1)}{(n!)^2} x^{n-\frac{\theta}{2}}.$$

On the other hand, again by (3.55), one easily observes that the mapping $b \mapsto \mathcal{M}_g(\frac{1}{2}+ib) \in \mathrm{L}^2(\mathbb{R})$ and therefore, by the Parseval identity of the Mellin transform, we have $g \in \mathrm{L}^2(\mathbb{R}_+)$, which further yields that

$$g^{(\theta)}(x) = x^{\frac{\theta}{2}} g(x) = \sum_{n=0}^{\infty} \frac{(-1)^n W_\phi(n+1)}{(n!)^2} x^n \in \mathrm{L}^2(\overline{m}).$$

Moreover, we recall from [17] that the law of I_ϕ is absolutely continuous, with a density denoted by ι, and is determined by its entire moments

$$\mathcal{M}_{I_\phi}(n+1) = \mathbb{E}[I_\phi^n] = \frac{n!}{\prod_{k=1}^n \phi(k)} = \frac{n!}{W_\phi(n+1)}, n \in \mathbb{N}. \tag{3.56}$$

Hence, by means of a standard application of Fubini theorem, see e.g. [113, Section 1.77], one shows that, for any $c, x > 0$,

$$\Lambda_\phi \mathrm{d}_c g^{(\theta)}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n W_\phi(n+1)}{(n!)^2} (cx)^n \mathcal{M}_{I_\phi}(n+1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}(cx)^n = \mathrm{d}_c\mathbf{e}(x),$$

where $\mathbf{e}(x) = e^{-x} \in \mathrm{L}^2(\overline{m})$. Since the span of $(\mathrm{d}_c\mathbf{e})_{c>0}$ is dense in $\mathrm{L}^2(\overline{m})$, we conclude that Λ_ϕ has a dense range in $\mathrm{L}^2(\overline{m})$. Next, combining (3.42) and (3.56), we obtain that, for all $n \in \mathbb{N}$,

$$\mathcal{M}_{V_\psi}(n+1)\mathcal{M}_{I_\phi}(n+1) = \frac{\prod_{k=1}^n (k-\theta)\phi(k)}{\prod_{k=1}^n \phi(k)} = \frac{\Gamma(n+1-\theta)}{\Gamma(1-\theta)} = \mathcal{M}_{G(1-\theta)}(n+1),$$

where we recall that $G(1-\theta)$ is a Gamma random variable with parameter $1-\theta$ whose law is denoted by m. Since both I_ϕ and $G(1-\theta)$ are moment determinate and so is V_ψ, see Theorem 3.4.1(1), we have

$$G(1-\theta) \stackrel{d}{=} V_\Psi \times I_\phi, \tag{3.57}$$

where $\stackrel{d}{=}$ stands for the identity in distribution and $\times$ represents the product of independent variables. Therefore, for any $f \in \mathrm{L}^2(m)$, by Hölder's inequality and the factorization identity (3.57), we have

$$\begin{aligned} \|\Lambda_\phi f\|_{\mathfrak{m}}^2 &\leq \int_0^\infty \Lambda_\phi f^2(x)\mathfrak{m}(x)dx = \int_0^\infty \int_0^\infty \iota(y) f^2(xy)dy\mathfrak{m}(x)dx &(3.58)\\ &= \int_0^\infty f^2(z)\int_0^\infty \frac{1}{x}\iota\left(\frac{z}{x}\right)\mathfrak{m}(x)dxdz = \int_0^\infty f^2(z)m(z)dz = \|f\|_m^2, &(3.59)\end{aligned}$$

where the second last equality comes from the factorization (3.57). Therefore, we see that $\Lambda_\phi \in \mathbf{B}(\mathrm{L}^2(m), \mathrm{L}^2(\mathfrak{m}))$ with $|||\Lambda_\phi||| \leq 1$. Next, for an arbitrary polynomial of order $n \in \mathbb{N}$, denoted by $\mathfrak{p}_n(x) = \sum_{i=0}^n a_i x^i, a_i \in \mathbb{R}$, we write $g_n(x) = \sum_{i=0}^n \frac{a_i}{\mathcal{M}_{V_\Psi}(i+1)} x^i$. It is easy to observe that $g_n \in \mathrm{L}^2(m)$ and $\Lambda_\phi g_n(x) = f_n(x)$. Therefore, $\mathfrak{p}_n \in \mathrm{Ran}(\Lambda_\phi) \subseteq \mathrm{L}^2(\mathfrak{m})$. Using the fact that V_ψ is moment determinate, we deduce that the set of polynomials are dense in $\mathrm{L}^2(\mathfrak{m})$, see [1, Corollary 2.3.3], hence Λ_ϕ has dense range in $\mathrm{L}^2(\mathfrak{m})$. Moreover, as Λ_ϕ is a Markov multiplier, i.e. $\Lambda_\phi \mathbf{1}(x) = \int_0^\infty \iota(y)dy = 1$ where here $\mathbf{1} = \mathbf{1}_{\mathbb{R}_+}$. Furthermore, observe that

$$\Lambda_\phi \mathbf{1}_{\{0\}}(x) = \int_0^\infty \iota(y)\mathbf{1}_{\{0\}}(xy)dy = \begin{cases} \int_0^\infty \iota(y)dy = 1 & \text{if} \quad x = 0, \\ 0 & \text{if} \quad x \neq 0, \end{cases}$$

and hence $\Lambda_\phi \mathbf{1}_{\{0\}} \equiv \mathbf{1}_{\{0\}}$. Moreover, for any $f \in \mathrm{L}^2(m)$, $\Lambda_\phi f(0) = \int_0^\infty f(0)\iota(y)dy = f(0)$. To prove similar results for $\widehat{\Lambda}_\phi$, let us first observe that for any $f \in \mathrm{L}^2(m), g \in \mathrm{L}^2(\mathfrak{m}), f, g \geq 0$,

$$\begin{aligned}\left\langle f, \widehat{\Lambda}_\phi g\right\rangle_m &= \left\langle \Lambda_\phi f, g\right\rangle_{\mathfrak{m}} = \int_0^\infty f(xy)\iota(y)dyg(x)\mathfrak{m}(x)dx \\ &= \int_0^\infty f(r)m^{-1}(r)\int_0^\infty \iota(r/x)g(x)\mathfrak{m}(x)/xm(r)dr \\ &= \int_0^\infty f(r)m^{-1}(r)\int_0^\infty g(rv)\mathfrak{m}(rv)\iota(1/v)1/vdvm(r)dr.\end{aligned}$$

Moreover, for any $f \in \mathrm{L}^2(m), g \in \mathrm{L}^2(\mathfrak{m}), |f| \in \mathrm{L}^2(m), |g| \in \mathrm{L}^2(\mathfrak{m})$, hence we get that for any $g \in \mathrm{L}^2(\mathfrak{m})$,

$$\widehat{\Lambda}_\phi g(x) \overset{a.e.}{=} \frac{1}{m(x)}\int_0^\infty g(xy)\mathfrak{m}(xy)\iota\left(\frac{1}{y}\right)\frac{1}{y}dy. \tag{3.60}$$

Therefore, for any $x \geq 0$, $\widehat{\Lambda}_\phi \mathbf{1}(x) = \frac{1}{m(x)}\int_0^\infty \mathfrak{m}(xy)\iota\left(\frac{1}{y}\right)\frac{1}{y}dy = 1$ by the factorization (3.57). Furthermore, both properties $\widehat{\Lambda}_\phi \mathbf{1}_{\{0\}} = \mathbf{1}_{\{0\}}$ and $\widehat{\Lambda}_\phi f(0) = f(0)$ can be proved using the same method as before. Next, we prove (3.43) in two steps. The first step is to establish (3.43) in $\mathrm{C}_0(\mathbb{R}_+)$. Note that by identities (3.39) and (3.32), in order to prove $P_t\Lambda_\phi = \Lambda_\phi Q_t$ on $\mathrm{C}_0(\mathbb{R}_+)$, it suffices to show only that $\overline{P}_t\Lambda_\phi = \Lambda_\phi \overline{Q}_t$ on $\mathrm{C}_0(\mathbb{R}_+)$, for which we use the criteria stated in [31, Proposition 3.2]. On the one hand, by (3.57), we have

$$\mathcal{M}_{G(1-\theta)}(z) = \mathcal{M}_{V_\Psi}(z)\mathcal{M}_{I_\phi}(z) \tag{3.61}$$

for all $z \in 1+i\mathbb{R}$. since $\mathcal{M}_{G(1-\theta)}(z) \neq 0$ on $z \in 1+i\mathbb{R}$ and $\mathcal{M}_{I_\phi}(z) < \infty$ on $z \in 1+i\mathbb{R}$, see [91, Proposition 6.7], we see from (3.61) that $\mathcal{M}_{V_\Psi}(z) \neq 0$ on $z = 1+i\mathbb{R}$. Hence by an application of the Wiener's Theorem, see e.g. [91, Lemma 7.9], one concludes that the multiplicative kernel $\mathcal{V}_\psi$ associated to V_Ψ, i.e. $\mathcal{V}_\psi f(x) = \mathbb{E}[f(xV_\Psi)]$, is injective on $\mathrm{C}_0(\mathbb{R}_+)$. This combined with (3.57) provides all conditions for the application of [31, Proposition 3.2], which gives that (3.43) holds for all $t \geq 0$ and $f \in \mathrm{C}_0(\mathbb{R}_+)$. Next, recalling that $\mathrm{C}_0(\mathbb{R}_+) \cap \mathrm{L}^2(\overline{m})$ is dense in $\mathrm{L}^2(\overline{m})$

(resp. $\mathrm{C}_0(\mathbb{R}_+) \cap \mathrm{L}^2(m)$ is dense in $\mathrm{L}^2(m)$), and since $\Lambda_\phi \in \mathbf{B}(\mathrm{L}^2(m), \mathrm{L}^2(\mathfrak{m}))$ and, for all $t \geq 0$, $\overline{P}_t \in \mathbf{B}(\mathrm{L}^2(\overline{m})), \overline{Q}_t \in \mathbf{B}(\mathrm{L}^2(\overline{m}))$ (resp. $P_t \in \mathrm{L}^2(\mathfrak{m}), Q_t \in \mathrm{L}^2(m)$), we conclude the extension of the intertwining relation between $\overline{P}$ and $\overline{Q}$ from $\mathrm{C}_0(\mathbb{R}_+)$ to $\mathrm{L}^2(\overline{m})$ (resp. between P and Q from $\mathrm{C}_0(\mathbb{R}_+)$ to $\mathrm{L}^2(m)$) by a density argument. Finally, using the properties of Λ_ϕ proved in the first statement, we can directly apply Theorem 3.2.1 to deduce (3.44) from (3.43). This concludes the proof of Theorem 3.4.1(3).

3.4.2 Proof of Theorem 3.4.1(4)

In order to compute $\Phi_{\overline{Y}}$, we first note that [94] has considered the normalization $\mathbb{E}_x[\tilde{\mathfrak{l}}_t^R] = \int_0^t \overline{q}_s(x,0)ds$, where $\overline{q}_s(x,y)$ is the transition density of $\overline{Q}$ with respect to the speed measure $\overline{m}$. Under this normalization, we have

$$c(\overline{m}) = \lim_{t\to 0} \frac{1}{t} \int_0^t \int_0^\infty \overline{m}(x)\overline{q}_s(x,0)dxds = 1$$

where we used the property that the integration of $\overline{q}_s(x,0)$ with respect to the speed measure is 1. Hence by [43, Section 5], we have, for $q > 0$,

$$\Phi_{\overline{Y}}(q) = 2\theta\tilde{\Phi}_R(q) = \frac{\Gamma(1-\theta)}{\Gamma(\theta)} 2^{1-\theta} q^\theta.$$

Combining this formula with the intertwining relation $\overline{P}_t\Lambda = \Lambda\overline{Q}_t$ and Theorem 3.2.1, we easily deduce that $\Phi_{\overline{X}} = \Phi_{\overline{Y}}$ and this completes proof of the first half of Theorem 3.4.1(4). Now let us focus on computing Φ_X and Φ_Y. As previously mentioned in Remark 3.4.1(ii), $\tilde{\mathfrak{l}}^Y$ is defined in [58] as the unique continuous increasing process such that

$$\mathbf{N}_t = Y_t^\theta - \tilde{\mathfrak{l}}_t^Y \quad \text{is a martingale,} \tag{3.62}$$

which uses the Doob-Meyer decomposition of the semi-martingale Y^θ, where we recall that Y is the squared radial Ornstein-Uhlenbeck process of order $-\theta$. The expression of $\tilde{\Phi}_Y$, the Laplace exponent of the inverse of $\tilde{l}^Y$, is given in (3.47). Therefore, our goal is to compute the constants $\tilde{c}(\mathfrak{m})$ and $\tilde{c}(m)$ and we simply have,

$$\Phi_X(q) = \frac{\tilde{\Phi}_X(q)}{\tilde{c}(\mathfrak{m})}, \quad \Phi_Y(q) = \frac{\tilde{\Phi}_Y(q)}{\tilde{c}(m)}.$$

In this direction, we will need the following Lemma, which is a generalization of [58, Proposition 2.1] from continuous semi-martingales to càdlàg semi-martingales, and serves as a stepping stone for computing $\tilde{c}(\mathfrak{m})$.

Lemma 3.4.1. *Let $(M_t)_{t\geq 0}$ be a càdlàg semi-martingale with $M_0 = 0$. Let $g : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing continuous function with $g(0) = 0$, and let $h : \mathbb{R}_+ \to \mathbb{R}_+$ be a strictly positive, continuous function, locally with bounded variation. We set*

$$N_t = h(t)M_{g(t)}, \quad t \geq 0,$$

and we denote by $\tilde{\mathfrak{l}}^M$ (resp. $\tilde{\mathfrak{l}}^N$) the local time at 0 of the càdlàg semi-martingale M (resp. N). Then $\tilde{\mathfrak{l}}^N$ can be obtained from a simple transform of $\tilde{\mathfrak{l}}^M$ by

$$\tilde{\mathfrak{l}}^N_t = \int_0^t h(s)d\tilde{\mathfrak{l}}^M_{g(s)}. \tag{3.63}$$

Proof. By definition of the local time via the Meyer-Tanaka formulae, see [96, Chapter IV], one has

$$\begin{aligned} |M_t| &= \int_0^t sgn(M_s)dM_s + \tilde{\mathfrak{l}}^M_t + \sum_{0<s\leq t}(|M_s| - |M_{s-}| - sgn(M_{s-})\Delta M_s), && (3.64)\\ |N_t| &= \int_0^t sgn(N_s)dN_s + \tilde{\mathfrak{l}}^N_t + \sum_{0<s\leq t}(|N_s| - |N_{s-}| - sgn(N_{s-})\Delta N_s), && (3.65) \end{aligned}$$

where the function sgn is the sign function defined by $sgn(x) = \mathbf{1}_{\{x>0\}} - \mathbf{1}_{\{x<0\}}$.

Consequently,

$$
\begin{aligned}
|M_{g(t)}| &= \int_0^{g(t)} sgn(M_s)dM_s + \tilde{\ell}^M_{g(t)} + \sum_{0<s\le g(t)} (|M_s| - |M_{s-}| - sgn(M_{s-})\Delta M_s) \\
&= \int_0^{t} sgn(N_s)d((h(s))^{-1}N_s) + \tilde{\ell}^M_{g(t)} + \sum_{0<s\le t} (h(s))^{-1}(|N_s| - |N_{s-}| - sgn(N_{s-})\Delta N_s) \\
&= \int_0^{t} sgn(N_s)(h(s))^{-1}dN_s - \int_0^t (h(s))^{-2}|N_s|dh(s) + \tilde{\ell}^M_{g(t)} \\
&\quad + \sum_{0<s\le t} (h(s))^{-1}(|N_s| - |N_{s-}| - sgn(N_{s-})\Delta N_s).
\end{aligned}
$$

Therefore using integration by parts, we have

$$
\begin{aligned}
|N_t| &= h(t)|M_{g(t)}| = \int_0^t h(s)dM_{g(s)} + \int_0^t M_{g(s)}dh(s) \\
&= \int_0^t sgn(N_s)d(N_s) - \int_0^t (h(s))^{-1}|N_s|dh(s) + \int_0^t h(s)d\tilde{\ell}^M_{g(s)} + \int_0^t (h(s))^{-1}|N_s|dh(s) \\
&\quad + \sum_{0<s\le t} (|N_s| - |N_{s-}| - sgn(N_{s-})\Delta N_s) \\
&= \int_0^t sgn(N_s)dN_s + \int_0^t h(s)d\tilde{\ell}^M_{g(s)} + \sum_{0<s\le t} (|N_s| - |N_{s-}| - sgn(N_{s-})\Delta N_s), \qquad (3.66)
\end{aligned}
$$

which, by identification between (3.65) and (3.66), yields that $\tilde{\ell}^N_t = \int_0^t h(s)d\tilde{\ell}^M_{g(s)}$.

□

Now let us compute the constants $\tilde{c}(\mathfrak{m})$ and $\tilde{c}(m)$. To this end, we first recall from [101] that $p_\theta(x) = x^\theta, x > 0$, is an invariant function for the semigroup $\overline{P}^\dagger$, therefore $\overline{P}_t p_\theta(x) \ge \overline{P}^\dagger_t p_\theta(x) = p_\theta(x)$, from which we deduce that the process $(\overline{X}^\theta) = (\overline{X}^\theta_t)_{t\ge 0}$ is a submartingale. Hence using a similar definition as (3.62), we define $\tilde{\ell}^{\overline{X}}$ as the unique increasing process such that

$$\mathbf{M}_t = \overline{X}^\theta_t - \tilde{\ell}^{\overline{X}}_t \quad \text{is a martingale.} \qquad (3.67)$$

Using the deterministic time change (3.31) between X and $\overline{X}$, we get $X^\theta_t =$

$e^{-\theta t}\overline{X}^{\theta}_{e^t-1}$, hence Lemma 3.4.1 yields that

$$\begin{aligned}\tilde{\mathfrak{l}}^X_t &= \int_0^t e^{-\theta s} d\tilde{\mathfrak{l}}^{\overline{X}}_{e^s-1} = \int_0^t e^{-\theta s}\left(d\overline{X}^{\theta}_{e^s-1} + d\mathbf{M}_{e^s-1}\right) = \int_0^t e^{-\theta s} d(e^{\theta s}X^{\theta}_s) + \int_0^t e^{-\theta s} d\mathbf{M}_{e^s-1} \\ &= \theta\int_0^t X^{\theta}_s ds + X^{\theta}_t - X^{\theta}_0 + \int_0^t e^{-\theta s} d\mathbf{M}_{e^s-1}.\end{aligned}$$

Now we observe that, on the one hand,

$$\begin{aligned}\int_0^\infty \mathbb{E}_x\left[\int_0^t X^{\theta}_s ds\right]\mathfrak{m}(x)dx = \int_0^t\int_0^\infty \mathbb{E}_x\left[X^{\theta}_s\right]\mathfrak{m}(x)dxds = \int_0^t \mathfrak{m}P_s p_\theta ds \\ = \int_0^t \mathfrak{m}p_\theta ds = \frac{W_\phi(1+\theta)}{\Gamma(1-\theta)\Gamma(1+\theta)},\end{aligned}$$

where we used the fact that $\mathfrak{m}(x)dx$ is an invariant measure for the semigroup P. On the other hand, by the martingale property of $(\mathbf{M}_t)_{t\geq 0}$, we have $\mathbb{E}_x[\int_0^t e^{-\theta s} d\mathbf{M}_{e^s-1}] = 0$ for all $x \geq 0$. Hence, by the definition of $\tilde{c}(\mathfrak{m})$, see (3.5), and the definition of semigroup P, we get

$$\begin{aligned}\tilde{c}(\mathfrak{m}) = \int_0^\infty \mathbb{E}_x[\tilde{\mathfrak{l}}^X_1]\mathfrak{m}(x)dx = \int_0^\infty \mathbb{E}_x\left[\left(\theta\int_0^1 X^{\theta}_s ds + X^{\theta}_1 - X^{\theta}_0\right)\right]\mathfrak{m}(x)dx \\ = \frac{\theta W_\phi(1+\theta)}{\Gamma(1-\theta)\Gamma(1+\theta)} + \mathfrak{m}P_1 p_\theta - \mathfrak{m}p_\theta = \frac{\theta W_\phi(1+\theta)}{\Gamma(1-\theta)\Gamma(1+\theta)}.\end{aligned}$$

In particular, since $\phi^Y(u) = u$, we have $\tilde{c}(m) = \frac{\theta}{\Gamma(1-\theta)}$, and Theorem 3.4.1(4) follows from dividing (3.47) by $\tilde{c}(m)$.

3.4.3 Proof of Theorem 3.4.1(5) and spectral expansions

In the section, we will prove Theorem 3.4.1(5) by providing the spectral expansion of $P_t f$ and $P^\dagger_t f$. In fact, we will find conditions on ψ, f and t such that these expansions hold. Note that the expansions for $\overline{P}$ and $\overline{P}^\dagger$ require additional analysis that will be detailed in a forthcoming paper, see already the paper by Patie and Zhao [93], which provides the spectral expansions for reflected stable processes. Let us start by recalling some well-known results for the self-adjoint semigroups Q and $Q^{\dagger}$. For $n \geq 0$, let $\mathcal{L}_n$ and $\mathcal{L}_n^{\dagger}$ be the Laguerre polynomials (of different orders) defined by

$$\mathcal{L}_n(x) = \frac{\mathcal{R}^{(n)} m(x)}{m(x)} = \sum_{k=0}^{n} (-1)^k \frac{\Gamma(n+1-\theta)}{\Gamma(k+1-\theta)\Gamma(n-k+1)} \frac{x^k}{k!}, \tag{3.68}$$

$$\mathcal{L}_n^{\dagger}(x) = \sum_{k=0}^{n} (-1)^k \frac{\Gamma(n+1+\theta)}{\Gamma(k+1+\theta)\Gamma(n-k+1)} \frac{x^{k+\theta}}{k!}, \tag{3.69}$$

where $\mathcal{R}^{(n)} f(x) = \frac{(x^n f(x))^{(n)}}{n!}$ is the Rodrigues operator. Then $\mathcal{L}_n \in \mathrm{L}^2(m)$ (resp. $\mathcal{L}_n^{\dagger} \in \mathrm{L}^2(m)$) is an eigenfunction of Q_t (resp. $Q_t^{\dagger}$) associated with eigenvalue e^{-nt} (resp. $e^{-(n+\theta)t}$), i.e. $Q_t \mathcal{L}_n(x) = e^{-nt} \mathcal{L}_n(x)$ (resp. $Q_t^{\dagger} \mathcal{L}_n^{\dagger}(x) = e^{-(n+\theta)t} \mathcal{L}_n^{\dagger}(x)$) for all $n \geq 0$. Moreover, for any $t > 0, f \in \mathrm{L}^2(m)$, Q_t and $Q_t^{\dagger}$ admit the following spectral expansions in $\mathrm{L}^2(m)$

$$Q_t f = \sum_{n=0}^{\infty} e^{-nt} \mathfrak{c}_n(-\theta) \langle f, \mathcal{L}_n \rangle_m \mathcal{L}_n, \tag{3.70}$$

$$Q_t^{\dagger} f = \frac{\Gamma(1-\theta)}{\Gamma(1+\theta)} \sum_{n=0}^{\infty} e^{-(n+\theta)t} \mathfrak{c}_n(\theta) \left\langle f, \mathcal{L}_n^{\dagger} \right\rangle_m \mathcal{L}_n^{\dagger}, \tag{3.71}$$

where for any $n \geq 0, u > -1$, we set

$$\mathfrak{c}_n(u) = \frac{\Gamma(1+u)\Gamma(n+1)}{\Gamma(n+1+u)}. \tag{3.72}$$

In order to study the spectral expansions of P and $P^{\dagger}$, we again recall from [101] that the function $p_\theta(x) = x^\theta$ is an invariant function for semigroup $\overline{P}^{\dagger}$. Hence we have

$$P_t^{\dagger} p_\theta(x) = \overline{P}^{\dagger}_{e^t-1} \mathrm{d}_{e^{-t}} p_\theta(x) = \overline{P}^{\dagger}_{1-e^{-t}} p_\theta(x e^{-t}) = p_\theta(x e^{-t}) = e^{-\theta t} p_\theta(x),$$

i.e. p_θ is a θ-invariant function for semigroup $P^{\dagger}$. Therefore, by Doob's h-transform, we can define a semigroup $P^{\uparrow} = (P_t^{\uparrow})_{t \geq 0}$, for $t \geq 0$ and $x > 0$, by

$$P_t^{\uparrow} f(x) = e^{\theta t} \frac{P_t^{\dagger} p_\theta f(x)}{p_\theta(x)}. \tag{3.73}$$

Note that $P^{\uparrow}$ is a generalized Laguerre semigroup associated to $\psi_{\uparrow} \in \mathcal{N}_{\uparrow}$, which we recall is defined as $\psi_{\uparrow}(u) = \psi(u+\theta)$ for all $u \geq 0$. Therefore, as shown in [91], the semigroup $P^{\uparrow}$ has an invariant measure $m^{\uparrow}$, whose law is absolutely continuous and determined by its entire moments

$$\mathcal{M}_{m^{\uparrow}}(n+1) = \frac{\prod_{k=1}^{n} \psi_{\uparrow}(k)}{n!}, n \in \mathbb{N}. \tag{3.74}$$

Next, we say that a sequence $(P_n)_{n\geq 0}$ in the Hilbert space $\mathrm{L}^2(\mathfrak{m})$ is a Bessel sequence if there exists $A > 0$ such that

$$\sum_{n=0}^{\infty} |\langle f, P_n\rangle_{\nu}|^2 \leq A\|f\|_{\nu}^2 \tag{3.75}$$

hold, for all $f \in \mathrm{L}^2(\mathfrak{m})$, see e.g. the monograph [34]. The constant A is called a Bessel bound. Recalling that the class $\mathcal{N}$ is defined as the collection of ψ in the form (3.34), we further define the following subclasses of $\mathcal{N}$. Denoting $\overline{\overline{\Pi}}(y) = \int_y^{\infty}\int_r^{\infty} \Pi(dx)dr$ the double tail of Π, we set

$$\mathcal{N}_P = \{\psi \in \mathcal{N}; \sigma^2 > 0\}, \tag{3.76}$$

$$\overline{\mathcal{N}}_{\infty} = \mathcal{N}_P \cup \{\psi \in \mathcal{N}; \sigma^2 = 0, \overline{\overline{\Pi}}(0+) = \infty\}. \tag{3.77}$$

Note that when $\psi \in \overline{\mathcal{N}}_{\infty}$ then $\lim_{u\to\infty} \frac{\psi(u)}{u} = \infty$. Moreover, define the following sets of (ψ, f),

$$\mathcal{D}^{\checkmark}(\Lambda_{\phi}) = \{(\psi, f);\ \psi \in \mathcal{N}_{\checkmark},\ f \in \mathrm{Ran}(\Lambda_{\phi})\}, \tag{3.78}$$

$$\mathcal{D}^{\mathcal{N}_P}(\mathfrak{m}) = \{(\psi, f);\ \psi \in \mathcal{N}_P \cap \mathcal{N}_{\checkmark},\ f \in \mathrm{L}^2(\mathfrak{m})\}. \tag{3.79}$$

Finally, for any $\psi \in \mathcal{N}$, we let

$$\mathcal{P}_n^{\psi}(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{k!}{\prod_{i=1}^{k} \psi(i)} x^k. \tag{3.80}$$

We are now ready to state the following theorem, which provides spectral properties of the non-self-adjoint semigroups $P_t f$ and $P_t^{\dagger} f$.

Theorem 3.4.2. *For any $\psi \in \mathcal{N}_{\checkmark}$, we have the following.*

1. *Let us write, for any $n \in \mathbb{N}$,*
$$\mathcal{P}_n(x) = \mathcal{P}_n^{\psi}(x), \quad \mathcal{P}_n^{\dagger}(x) = x^{\theta}\mathcal{P}_n^{\psi_{\uparrow}}(x). \tag{3.81}$$
Then $\mathcal{P}_n \in \mathrm{L}^2(\mathfrak{m})$ (resp. $\mathcal{P}_n^{\dagger} \in \mathrm{L}^2(\mathfrak{m})$) is an eigenfunction of P_t (resp. $P_t^{\dagger}$) associated to the eigenvalue e^{-nt} (resp. $e^{-(n+\theta)t}$). Moreover, the sequence $\left(\mathfrak{c}_n^{-\frac{1}{2}}(-\theta)\mathcal{P}_n\right)_{n\geq 0}$ is a dense Bessel sequence in $\mathrm{L}^2(\mathfrak{m})$ with upper bound 1, where we recall that $\mathfrak{c}_n(u)$ is defined in (3.72)*. Finally, we have $(e^{-nt})_{n\geq 0} = S(Q_t) \subseteq S(P_t)$, and $(e^{-(n+\theta)t})_{n\geq 0} = S(Q_t^{\dagger}) \subseteq S(P_t^{\dagger})$.*

2. *For any $\psi \in \mathcal{N}_{\checkmark} \cap \overline{\mathcal{N}}_{\infty}$ and $n \geq 0$, let*
$$\mathfrak{m}_n(x) = \frac{\mathcal{R}^{(n)}\mathfrak{m}(x)}{\mathfrak{m}(x)}, \quad \mathfrak{m}_n^{\dagger}(x) = \frac{\mathcal{R}^{(n)}m^{\uparrow}(x)}{x^{\theta}\mathfrak{m}(x)}. \tag{3.82}$$
Then $\mathfrak{m}_n$ (resp. $\mathfrak{m}_n^{\dagger}$) is an eigenfunction of $\widehat{P}_t$ (resp. $\widehat{P}_t^{\dagger}$) associated to the eigenvalue e^{-nt} (resp. $e^{-(n+\theta)t}$). Moreover, the sequences $(\mathcal{P}_n)_{n\geq 0}$ and $(\mathfrak{m}_n)_{n\geq 0}$ (resp. $(\mathcal{P}_n^{\dagger})_{n\geq 0}$ and $(\mathfrak{m}_n^{\dagger})_{n\geq 0}$) are biorthogonal sequences in $\mathrm{L}^2(\mathfrak{m})$. Furthermore, if $\psi \in \mathcal{N}_P \cap \mathcal{N}_{\checkmark}$, then for any $\epsilon > 0$ and large n,
$$\|\mathfrak{m}_n\|_{\mathfrak{m}} = \mathrm{O}(e^{\epsilon n}). \tag{3.83}$$
If in addition $\overline{\overline{\Pi}}(0+) < \infty$, then with $\mathfrak{b} = \frac{\beta + \overline{\overline{\Pi}}(0+)}{\sigma^2}$, we have for large n,
$$\|\mathfrak{m}_n\|_{\mathfrak{m}} = \mathrm{O}(n^{\mathfrak{b}}), \tag{3.84}$$
and the sequence $(\sqrt{\mathfrak{c}_n(\mathfrak{b})}\mathfrak{m}_n)_{n\geq 0}$ is a Bessel sequence in $\mathrm{L}^2(\mathfrak{m})$ with bound 1.

3. *For any $t > 0$ and $(\psi, f) \in \mathcal{D}^{\checkmark}(\Lambda_{\phi}) \cup \mathcal{D}^{\mathcal{N}_P}(\mathfrak{m})$, we have in $\mathrm{L}^2(\mathfrak{m})$ the following spectral expansions*
$$P_t f(x) = \sum_{n=0}^{\infty} e^{-nt} \langle f, \mathfrak{m}_n \rangle_{\mathfrak{m}} \mathcal{P}_n(x), \tag{3.85}$$
$$P_t^{\dagger} f(x) = \sum_{n=0}^{\infty} e^{-(n+\theta)t} \left\langle f, \mathfrak{m}_n^{\dagger} \right\rangle_{\mathfrak{m}} \mathcal{P}_n^{\dagger}(x). \tag{3.86}$$

Before proving the previous Theorem, we state the following corollary which gives the speed of convergence to equilibrium in the Hilbert space topology $\mathrm{L}^2(\mathfrak{m})$.

Corollary 3.4.1. *Let $\psi \in \mathcal{N}_P \cap \mathcal{N}_{\checkmark}$ with $\overline{\overline{\Pi}}(0+) < \infty$, then recalling that $\mathfrak{b} = \frac{\beta + \overline{\overline{\Pi}}(0+)}{\sigma^2}$, we have, for any $f \in \mathrm{L}^2(\mathfrak{m})$ and $t > 0$,*

$$\|P_t f - \mathfrak{m} f\|_{\mathfrak{m}} \leq \sqrt{\frac{\mathfrak{b}+1}{1-\theta}} e^{-t} \|f - \mathfrak{m} f\|_{\mathfrak{m}}. \tag{3.87}$$

The rest of this section is devoted to the proof of these results.

Proof of Theorem 3.4.2(1)

Let $\psi \in \mathcal{N}_{\checkmark}$ and recall that $\Lambda_\phi p_k(x) = \mathbb{E}[x^k I_\phi^k] = \frac{k!}{a_k(\phi)} p_k(x)$. Use the linearity of Λ_ϕ and note that for any $n \geq 0$,

$$\begin{aligned}
\Lambda_\phi \mathcal{L}_n(x) &= \sum_{k=0}^{n} \frac{(-1)^k \Gamma(n+1-\theta)}{\Gamma(k+1-\theta)\Gamma(n-k+1)} \frac{1}{k!} \Lambda_\phi p_k(x) \\
&= \sum_{k=0}^{n} \frac{(-1)^k (n-\theta)\ldots(k+1-\theta)}{(n-k)!} \frac{1}{\prod_{i=1}^k \phi(i)} p_k(x) \\
&= \sum_{k=0}^{n} (-1)^k \frac{(n-\theta)\ldots(k+1-\theta)}{(n-k)!} \prod_{i=1}^{k} \frac{i-\theta}{\psi(i)} p_k(x) = \binom{n-\theta}{n} \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k k!}{\prod_{i=1}^k \psi(i)} x^k \\
&= \frac{\mathcal{P}_n(x)}{\mathfrak{c}_n(-\theta)}.
\end{aligned}$$

Since $\mathcal{L}_n \in \mathrm{L}^2(m)$, and $\Lambda_\phi \in \mathbf{B}(\mathrm{L}^2(m), \mathrm{L}^2(\mathfrak{m}))$, we get that $\mathcal{P}_n \in \mathrm{L}^2(\mathfrak{m})$. Apply the intertwining relation (3.43), together with $Q_t \mathcal{L}_n(x) = e^{-nt} \mathcal{L}_n(x)$, we get, for each $n \in \mathbb{N}$,

$$P_t \mathcal{P}_n(x) = \mathfrak{c}_n(-\theta) P_t \Lambda_\phi \mathcal{L}_n(x) = \mathfrak{c}_n(-\theta) \Lambda_\phi Q_t \mathcal{L}_n(x) = \mathfrak{c}_n(-\theta) e^{-nt} \Lambda_\phi \mathcal{L}_n(x) = e^{-nt} \mathcal{P}_n(x).$$

This proves the eigenfunction property of $\mathcal{P}_n$. Next, using the fact that V_ψ is moment determinate, we see that the set of polynomials are dense in $\mathrm{L}^2(\mathfrak{m})$, see

[1, Corollary 2.3.3], which proves the completeness of $(\mathcal{P}_n)_{n\geq 0}$. Next, to get the Bessel property of $\left(\mathfrak{c}_n^{-\frac{1}{2}}(-\theta)\mathcal{P}_n\right)_{n\geq 0}$, we observe that, for any $f \in \mathrm{L}^2(\mathfrak{m})$,

$$\sum_{n=0}^{\infty}\left|\left\langle f, \mathfrak{c}_n^{-\frac{1}{2}}(-\theta)\mathcal{P}_n\right\rangle_{\mathfrak{m}}\right|^2 = \sum_{n=0}^{\infty}\left|\left\langle f, \sqrt{\mathfrak{c}_n(-\theta)}\Lambda_\phi\mathcal{L}_n\right\rangle_{\mathfrak{m}}\right|^2 = \sum_{n=0}^{\infty}\left|\left\langle \widehat{\Lambda}_\phi f, \sqrt{\mathfrak{c}_n(-\theta)}\mathcal{L}_n\right\rangle_{\mathfrak{m}}\right|^2$$
$$= \|\widehat{\Lambda}_\phi f\|_m^2 \leq \|f\|_{\mathfrak{m}}^2,$$

where we used the Parseval identity for the (normalized) Laguerre polynomials in $\mathrm{L}^2(m)$, see e.g. [7, Section 2.7], and the fact that $\widehat{\Lambda}_\phi \in \mathbf{B}(\mathrm{L}^2(\mathfrak{m}), \mathrm{L}^2(m))$ as the adjoint of $\Lambda_\phi \in \mathbf{B}(\mathrm{L}^2(m), \mathrm{L}^2(\mathfrak{m}))$ with $|||\widehat{\Lambda}_\phi||| = |||\Lambda_\phi||| \leq 1$. Finally, using similar computations than above, we observe that $\mathcal{P}_n^\dagger = \frac{W_\phi(1+\theta)}{\Gamma(1+\theta)}\mathfrak{c}_n(\theta)\Lambda_\phi\mathcal{L}_n^\dagger$, and the proof for $\mathcal{P}_n^\dagger$ being an eigenfunction for $P_t^\dagger$ with eigenvalue $e^{-(n+\theta)t}$ follows through a similar line of reasoning using the intertwining relation with $Q_t^\dagger$. This concludes the proof.

Proof of Theorem 3.4.2 (2)

Let us write $\mathcal{T}_1\psi(u) = \frac{u}{u+1}\psi(u+1)$ for $u > 0$, then by [68, Lemma 2.1], $\mathcal{T}_1\psi$ is the Laplace exponent of a spectrally negative Lévy process, which satisfies $\mathcal{T}_1\psi(0) = 0$ and $(\mathcal{T}_1\psi)'(0) = \psi(1) > 0$. Hence $\mathcal{T}_1\psi \in \mathcal{N}_\uparrow$ and therefore by [91, Theorem 1.5], $\mathcal{T}_1\psi$ characterizes a generalized Laguerre semigroup, denoted by $\check{P} = (\check{P}_t)_{t\geq 0}$, with an invariant measure denoted by $\check{\mathfrak{m}}$, and the spectral properties of $\check{P}$ have been studied in [91]. In the rest of the paper, this semigroup $\check{P}$ will serve as a reference semigroup in order for us to develop further spectral results for P. Our first aim is to establish an intertwining relation between the semigroups P and $\check{P}$. To this end, we need introduce a few objects and notation. Let $\mathtt{Z}$ be a random variable whose law is given by

$$\mathbb{P}(\mathtt{Z} \in dx) = \psi(1)W_+'(-\ln x)dx + W(0)\delta_1(x), \quad x \in [0,1], \tag{3.88}$$

with δ_1 denoting the Dirac mass at 1, and W'_+ being the right-derivative of the so-called scale function of the Lévy process ξ, see e.g. [67, Section 8.2], which is an increasing function $W : [0,\infty) \longrightarrow [0,\infty)$ characterized by its Laplace transform

$$\int_0^\infty e^{-\lambda x} W(x)dx = \frac{1}{\psi(\lambda)}, \quad \lambda > 0. \tag{3.89}$$

We also recall that $W(0) = 0$ whenever $\psi \in \overline{\mathcal{N}}_\infty$ and thus in such case the law of Z is absolutely continuous with a density denoted by $\mathfrak{z}$. We are now ready to state and prove the following lemma.

Lemma 3.4.2. *Define the multiplicative kernel Λ_{Z} as $\Lambda_{\mathrm{Z}} f(x) = \mathbb{E}[f(x\mathrm{Z})]$, then $\Lambda_{\mathrm{Z}} \in \mathbf{B}(\mathrm{C}_0(\mathbb{R}_+)) \cap \mathbf{B}(\mathrm{L}^2(\mathfrak{m}), \mathrm{L}^2(\check{\mathfrak{m}}))$ with $|||\Lambda_{\mathrm{Z}}||| \leq 1$. Furthermore, for all $f \in \mathrm{L}^2(\mathfrak{m})$, we have*

$$\Lambda_{\mathrm{Z}} P_t f = \check{P}_t \Lambda_{\mathrm{Z}} f. \tag{3.90}$$

Proof. First, we observe that, for all $n \in \mathbb{N}$,

$$\mathcal{M}_{V_\Psi}(n+1) = \frac{\prod_{k=1}^n \psi(k)}{n!} = \frac{\prod_{k=1}^n \frac{k}{k+1}\psi(k+1)}{n!}\frac{\psi(1)(n+1)}{\psi(n+1)} = \mathcal{M}_{V_{\mathcal{T}_1\psi}}(n+1)\frac{\psi(1)(n+1)}{\psi(n+1)},$$

where, by [91, Theorem 2.1], $V_{\mathcal{T}_1\psi}$ is the random variable whose law is the stationary distribution of $\check{P}$ and is determined by its entire moments $\mathcal{M}_{V_{\mathcal{T}_1\psi}}(n+1) = \frac{\prod_{k=1}^n \mathcal{T}_1\psi(k)}{n!}$. Now by (3.89), we have, using an obvious change of variable and integration by parts, that for each $n \in \mathbb{N}$,

$$\frac{1}{\psi(n+1)} = \int_0^\infty e^{-(n+1)x} W(x)dx = \int_0^1 u^n W(-\ln u)du = \frac{1}{n+1}\left(W(0) + \int_0^1 u^n W'_+(-\ln u)du\right).$$

Therefore,

$$\begin{aligned}\mathcal{M}_{V_\Psi}(n+1) &= \mathcal{M}_{V_{\mathcal{T}_1\psi}}(n+1)\frac{\psi(1)(n+1)}{\psi(n+1)} = \mathcal{M}V_{\mathcal{T}_1\psi}(n+1)\psi(1)\int_0^1 u^n W'_+(-\ln u) + W(0)\delta_1(u)du \\ &= \mathcal{M}_{V_{\mathcal{T}_1\psi}}(n+1)\int_0^1 u^n \zeta(u)du = \mathcal{M}_{V_{\mathcal{T}_1\psi}}(n+1)\mathcal{M}_{\mathrm{Z}}(n+1).\end{aligned}$$

Both variables V_Ψ and $V_{\mathcal{T}_1\psi}$ are moment determinate by Theorem 3.4.1(1) and [91, Theorem 2.1], and so does Z since it has compact support. Hence we conclude that

$$V_\Psi \stackrel{d}{=} V_{\mathcal{T}_1\psi} \times \mathsf{Z}. \tag{3.91}$$

Therefore, the facts that $\Lambda_{\mathsf{Z}} \in \mathbf{B}(\mathrm{L}^2(\mathfrak{m}), \mathrm{L}^2(\breve{\mathfrak{m}}))$ and $|||\Lambda_{\mathsf{Z}}||| \leq 1$ follow from similar arguments as (3.58) and $\Lambda_{\mathsf{Z}} \in \mathbf{B}(\mathrm{C}_0(\mathbb{R}_+))$ follows easily from dominated convergence. Moreover, by [91, Lemma 7.9], the multiplicative kernel $\mathcal{V}_{\mathcal{T}_1\psi}$ defined by $\mathcal{V}_{\mathcal{T}_1\psi} f(x) = \mathbb{E}[f(xV_{\mathcal{T}_1\psi})]$ is one-to-one in $\mathrm{C}_0(\mathbb{R}_+)$. Hence again using [31, Proposition 3.2], the intertwining relation (3.90) holds for all $f \in \mathrm{C}_0(\mathbb{R}_+)$, and we can further extend this relation to $\mathrm{L}^2(\mathfrak{m})$ using a density argument as $\mathrm{C}_0(\mathbb{R}_+) \cap \mathrm{L}^2(\mathfrak{m})$ is dense in $\mathrm{L}^2(\mathfrak{m})$ and the fact that $P_t \in \mathrm{L}^2(\mathfrak{m})$, $\breve{P}_t \in \mathrm{L}^2(\breve{\mathfrak{m}})$. This completes the proof. □

Corollary 3.4.2. *For any $\psi \in \overline{\mathcal{N}}_\infty \cap \mathcal{N}_\checkmark$, we have $\mathfrak{m}(x) > 0$ for any $x > 0$ and $\mathfrak{m} \in \mathrm{C}^\infty(\mathbb{R}_+)_0(\mathbb{R}_+)$.*

Proof. Let us write $\phi_1(u) = \frac{\mathcal{T}_1\psi(u)}{u}, u \geq 0$, then since $\mathcal{T}_1\psi \in \mathcal{N}_\uparrow$, an application of the Wiener-Hopf factorization yields that ϕ_1 is a Bernstein function, see [91, (1.8)]. Moreover, by observing that $\phi_1(u) = \frac{u+1-\theta}{u+1}\phi(u+1)$, it is easy to see that $\lim_{u\to\infty} \phi_1(u) = \phi(u) = \infty$ as $\psi \in \overline{\mathcal{N}}_\infty$. Hence by [91, Theorem 1.6], the density of $\breve{\mathfrak{m}}$ is concentrated and positive on $(0, \infty)$. Now since, for all $n \in \mathbb{N}$

$$\mathbb{E}[V_\psi^{n+1}] = \frac{\prod_{k=1}^{n+1}\psi(k)}{(n+1)!} = \psi(1)\frac{\prod_{k=1}^{n}\mathcal{T}_1\psi(k)}{n!} = \psi(1)\mathbb{E}[V_{\mathcal{T}_1\psi}^n],$$

we get by moment determinacy that

$$x\mathfrak{m}(x) = \psi(1)\breve{\mathfrak{m}}(x), \quad x > 0. \tag{3.92}$$

This implies that the density of $\mathfrak{m}$ has the same support as $\breve{\mathfrak{m}}$. Now let Π_1 denote the Lévy measure of $\mathcal{T}_1\psi$, then by [85, Theorem 2.2],

$$\overline{\Pi}_1(y) = \int_y^\infty (e^{-r}\overline{\Pi}(r)dr + e^{-r}\Pi(dr)) = e^{-y}\overline{\Pi}(y), \quad \overline{\Pi}_1(0+) = \overline{\Pi}(0+), \tag{3.93}$$

therefore if $\psi \in \overline{\mathcal{N}}_\infty$, so does $\mathcal{T}_1\psi$ and therefore $\breve{\mathfrak{m}} \in C_0^\infty(\mathbb{R}_+)$ by [91, Theorem 2.5]. Again using (3.92), $\mathfrak{m}$ and $\breve{\mathfrak{m}}$ have the same smoothness properties, which shows that $\mathfrak{m} \in C_0^\infty(\mathbb{R}_+)$. □

We now have all the ingredients to prove Theorem 3.4.2(2). From (3.93), it is easy to see that if $\psi \in \overline{\mathcal{N}}_\infty \cap \mathcal{N}_\checkmark$, then $\mathcal{T}_1\psi \in \overline{\mathcal{N}}_\infty \cap \mathcal{N}_\uparrow$ and we see from [91, Theorem 2.19] that $\breve{P}_t$ has co-eigenfunctions $\breve{\mathfrak{m}}_n \in L^2(\breve{\mathfrak{m}})$, given by $\breve{\mathfrak{m}}_n(x) = \frac{\mathcal{R}^{(n)}\breve{\mathfrak{m}}(x)}{\breve{\mathfrak{m}}(x)}$. Now let us define, for any $n \in \mathbb{N}$,

$$\mathfrak{m}_n = \widehat{\Lambda}_Z\breve{\mathfrak{m}}_n, \tag{3.94}$$

then $\mathfrak{m}_n \in L^2(\mathfrak{m})$ since $\widehat{\Lambda}_Z \in \mathbf{B}(L^2(\breve{\mathfrak{m}}), L^2(\mathfrak{m}))$. Moreover, similar to (3.60), we deduce that, for almost every (a.e.) $x > 0$,

$$\mathfrak{m}_n(x) = \widehat{\Lambda}_Z\breve{\mathfrak{m}}_n(x) = \frac{1}{\mathfrak{m}(x)}\int_0^\infty y^{-1}\breve{\mathfrak{m}}_n(xy)\breve{\mathfrak{m}}(xy)\mathfrak{z}\left(\frac{1}{y}\right)dy = \frac{1}{\mathfrak{m}(x)}\int_0^\infty y^{-1}\mathcal{R}^{(n)}\breve{\mathfrak{m}}(xy)\mathfrak{z}\left(\frac{1}{y}\right)dy,$$

where we recall that $\mathfrak{z}$ denotes the density of the random variable Z whose law is absolutely continuous as $W(0) = 0$ with $\psi \in \overline{\mathcal{N}}_\infty$. We write, for any $n \in \mathbb{N}$, $w_n(x) = \mathfrak{m}_n(x)\mathfrak{m}(x)$ and $\breve{w}_n(x) = \breve{\mathfrak{m}}_n(x)\breve{\mathfrak{m}}(x) = \mathcal{R}^{(n)}\breve{\mathfrak{m}}(x), x > 0$, then the above equation is equivalent to

$$w_n(x) = \int_0^\infty y^{-1}\breve{w}_n(xy)\mathfrak{z}\left(\frac{1}{y}\right)dy \tag{3.95}$$

for a.e. $x > 0$. In other words, we have, with the obvious notation, $w_n \overset{a.e.}{=} \breve{w}\surd\mathfrak{z}$ where $\surd$ represents the Mellin convolution, see [77, Section 11.11]. Therefore, by [77, (11.11.4)], we have, for any $\Re(z) > n$,

$$\mathcal{M}_{w_n}(z) = \mathcal{M}_Z(z)\mathcal{M}_{\breve{w}_n}(z) = \mathcal{M}_Z(z)\frac{(-1)^n}{n!}\frac{\Gamma(z)}{\Gamma(z-n)}\mathcal{M}_{V_{\mathcal{T}_1\psi}}(z) = \frac{(-1)^n}{n!}\frac{\Gamma(z)}{\Gamma(z-n)}\mathcal{M}_{V_\psi}(z)$$

where the last identity comes from the factorization (3.91). Observe that the right-hand side of the above equation is indeed the Mellin transform of $\mathcal{R}^{(n)}\mathfrak{m}(x)$, and by injectivity of the Mellin transform, we conclude that $w_n(x) \overset{a.e.}{=} \mathcal{R}^{(n)}\mathfrak{m}(x)$, or equivalently

$$\mathfrak{m}_n(x) = \frac{\mathcal{R}^{(n)}\mathfrak{m}(x)}{\mathfrak{m}(x)}$$

for almost every $x > 0$, which can be extended to every $x > 0$ by the continuity of $\mathfrak{m}_n$ and the smoothness of $\mathfrak{m}$, see Corollary 3.4.2. Furthermore, by the intertwining relationship (3.90),

$$\widehat{P}_t\mathfrak{m}_n(x) = \widehat{P}_t\widehat{\Lambda}_{\mathrm{Z}}\check{\mathfrak{m}}_n(x) = \widehat{\Lambda}_{\mathrm{Z}}\widehat{\check{P}}_t\check{\mathfrak{m}}_n(x) = e^{-nt}\widehat{\Lambda}_{\mathrm{Z}}\check{\mathfrak{m}}_n(x) = e^{-nt}\mathfrak{m}_n(x), \tag{3.96}$$

which shows that $\mathfrak{m}_n$ is an eigenfunction for $\widehat{P}$ (or co-eigenfunction for P). Finally, take any $g \in \mathrm{L}^2(m)$, then by the co-eigenfunction property of $\mathfrak{m}_n$ and the intertwining relation (3.43), we have

$$\begin{aligned} e^{-nt}\left\langle \widehat{\Lambda}_\phi\mathfrak{m}_n, g\right\rangle_m &= e^{-nt}\left\langle \mathfrak{m}_n, \Lambda_\phi g\right\rangle_{\mathfrak{m}} = \left\langle \widehat{P}_t\mathfrak{m}_n, \Lambda_\phi g\right\rangle_{\mathfrak{m}} = \left\langle \mathfrak{m}_n, P_t\Lambda_\phi g\right\rangle_{\mathfrak{m}} \\ &= \left\langle \mathfrak{m}_n, \Lambda_\phi Q_t g\right\rangle_{\mathfrak{m}} = \left\langle \widehat{\Lambda}_\phi\mathfrak{m}_n, Q_t g\right\rangle_m . \end{aligned}$$

In other words, $\widehat{\Lambda}_\phi\mathfrak{m}_n$ is a co-eigenfunction of Q_t, which is indeed $\mathcal{L}_n$ since Q_t is self-adjoint. Moreover, recalling that Λ_ϕ has a dense range in $\mathrm{L}^2(\mathfrak{m})$, we have that $\widehat{\Lambda}_\phi$ is one-to-one on $\mathrm{L}^2(\mathfrak{m})$ and thus equation $\widehat{\Lambda}_\phi f = \mathcal{L}_n$ has at most one solution in $\mathrm{L}^2(\mathfrak{m})$, which is indeed $\mathfrak{m}_n$. Therefore, we deduce that, for any $m, n \geq 0$,

$$\langle \mathcal{P}_m, \mathfrak{m}_n\rangle_{\mathfrak{m}} = \mathfrak{c}_m(-\theta)\left\langle \Lambda_\phi\mathcal{L}_m, \mathfrak{m}_n\right\rangle_{\mathfrak{m}} = \mathfrak{c}_m(-\theta)\left\langle \mathcal{L}_m, \widehat{\Lambda}_\phi\mathfrak{m}_n\right\rangle_m = \mathfrak{c}_m(-\theta)\left\langle \mathcal{L}_m, \mathcal{L}_n\right\rangle_m = \mathbf{1}_{\{m=n\}}, \tag{3.97}$$

by the orthogonality property of the Laguerre polynomials. This shows that the sequences $(\mathcal{P}_n)_{n\geq 0}$ and $(\mathfrak{m}_n)_{n\geq 0}$ are biorthogonal. Next, by [33], $\mathcal{T}_1\psi$ and ψ have the same parameter σ^2, hence $\psi \in \mathcal{N}_P \cap \mathcal{N}_{\checkmark}$ if and only if $\mathcal{T}_1\psi \in \mathcal{N}_P \cap \mathcal{N}_{\uparrow}$. Moreover, observing that $\phi(\infty) = \phi_1(\infty) = \beta + \overline{\overline{\Pi}}(0+)$, hence by [91, Theorem 9.1

and Theorem 10.1], the bounds on the right-hand side of (3.83) and (3.84) hold for $\|\breve{\mathfrak{m}}_n\|_{\breve{\mathfrak{m}}}$. Since $\mathfrak{m}_n = \widehat{\Lambda}_{\mathrm{Z}}\breve{\mathfrak{m}}_n$ and $|||\widehat{\Lambda}_{\mathrm{Z}}||| = |||\Lambda_{\mathrm{Z}}||| \leq 1$, we conclude the same bounds for $\|\mathfrak{m}_n\|_{\mathfrak{m}}$. Finally, by [91, Theorem 10.1], the sequence $(\sqrt{\mathfrak{c}_n(\mathfrak{b})}\breve{\mathfrak{m}}_n)_{n\geq 0}$ is a Bessel sequence in $\mathrm{L}^2(\breve{\mathfrak{m}})$ with bound 1, hence we have, for any $f \in \mathrm{L}^2(\mathfrak{m})$,

$$\sum_{n=0}^{\infty}\left|\left\langle f,\ \sqrt{\mathfrak{c}_n(\mathfrak{b})}\mathfrak{m}_n\right\rangle_{\mathfrak{m}}\right|^2 = \sum_{n=0}^{\infty}\left|\left\langle f,\ \sqrt{\mathfrak{c}_n(\mathfrak{b})}\widehat{\Lambda}_{\mathrm{Z}}\breve{\mathfrak{m}}_n\right\rangle_{\mathfrak{m}}\right|^2 = \sum_{n=0}^{\infty}\left|\left\langle \Lambda_{\mathrm{Z}}f,\ \sqrt{\mathfrak{c}_n(\mathfrak{b})}\breve{\mathfrak{m}}_n\right\rangle_{\breve{\mathfrak{m}}}\right|^2 \leq \|\Lambda_{\mathrm{Z}}f\|^2_{\breve{\mathfrak{m}}} \leq \|f\|^2_{\mathfrak{m}}$$

since $|||\Lambda_{\mathrm{Z}}||| \leq 1$. This proves that $(\sqrt{\mathfrak{c}_n(\mathfrak{b})}\mathfrak{m}_n)_{n\geq 0}$ is a Bessel sequence in $\mathrm{L}^2(\mathfrak{m})$. Now in the case of $\mathfrak{m}_n^{\dagger}$, let us first prove that it is in $\mathrm{L}^2(\mathfrak{m})$, which suffices to show its $\mathrm{L}^2(\mathfrak{m})$-integrability around the neighborhoods of 0 and infinity. To this end, define $d_{\phi_1} = \sup\{u < 0; \phi_1(u) = -\infty \text{ or } \phi_1(u) = 0\}$, where we recall that $\phi_1(u) = \frac{\mathcal{T}_1\psi(u)}{u} = \frac{\psi(u+1)}{u+1}$, then we easily observe that $d_{\phi_1} = \theta - 1$ since θ is the largest root of ψ. Hence by combining [91, Theorem 5.4] and (3.92), we see that for any $a > \theta$ and $A \in (0, \mathfrak{r})$, that exists a constant $C_{a,A} > 0$ such that $\mathfrak{m}(x) \geq C_{a,A}x^a$ for all $x \in (0, A)$. Therefore, denoting $w_n^{\dagger} = \mathfrak{m}_n^{\dagger}\mathfrak{m}$, then we see that

$$(\mathfrak{m}_n^{\dagger}(x))^2\mathfrak{m}(x) = \frac{(w_n^{\dagger}(x))^2}{\mathfrak{m}(x)} \leq \frac{1}{C_{a,A}}x^{-a}(w_n^{\dagger}(x))^2$$

for all $x \in (0, A)$. Hence to prove the $\mathrm{L}^2(\mathfrak{m})$-integrability of $\mathfrak{m}_n^{\dagger}$ around 0, it suffices to prove the $\mathrm{L}^2(p_{-a})$-integrability of $w_n^{\dagger}$ around 0, where $p_{-a}(x)dx = x^{-a}dx$. However, observe that $w_n^{\dagger} = \frac{\mathcal{R}^{(n)}m^{\uparrow}}{p_{\theta}}$, thus by taking the Mellin transform on both sides, we have, for $\Re(z) > n + \theta$,

$$\mathcal{M}_{w_n^{\dagger}}(z) = \mathcal{M}_{\mathcal{R}^{(n)}m^{\uparrow}}(z-\theta) = \frac{(-1)^n}{n!}\frac{\Gamma(z-\theta)}{\Gamma(z-\theta-n)}W_{\phi^{\uparrow}}(z-\theta) = \frac{(-1)^n}{n!}\frac{\Gamma(z-\theta)}{\Gamma(z-\theta-n)}\frac{W_{\phi}(z)}{W_{\phi}(1+\theta)},$$

where for the last identity we used [91, (8.12)], with $\phi^{\uparrow}(u) = \frac{\psi_{\uparrow}(u)}{u} = \phi(u+\theta)$. Therefore, using the Stirling approximation (3.54) as well as the asymptotic behavior of W_{ϕ} by [91, Theorem 5.1(3)], we have, for large $|b|$, that

$$\mathcal{M}_{p_{-\frac{a}{2}}w_n^{\dagger}}\left(\frac{1}{2} + ib\right) = \mathcal{M}_{w_n^{\dagger}}\left(\frac{1-a}{2} + ib\right) = \mathrm{o}\left(|b|^{n-u}\right) \tag{3.98}$$

for some $u > n+\frac{1}{2}$. Hence $b \mapsto \mathcal{M}_{p_{\frac{-a}{2}} w_n^\dagger}\left(\frac{1}{2}+ib\right) \in \mathrm{L}^2(\mathbb{R})$, and $x \mapsto x^{-\frac{a}{2}} w_n^\dagger(x) \in \mathrm{L}^2(\mathbb{R}_+)$ by the Parseval identity of Mellin transform, that is $w_n^\dagger \in \mathrm{L}^2(p_{-a})$. This proves the $\mathrm{L}^2(\mathfrak{m})$-integrability of $\mathfrak{m}_n^\dagger$ around 0. On the other hand, since $\mathcal{M}_{m^\uparrow}(u) = W_{\phi^\uparrow}(u) = \frac{W_\phi(u+\theta)}{W_\phi(1+\theta)}$, we have

$$\mathcal{M}_{p_\theta \mathfrak{m}}(u) = \mathcal{M}_{\mathfrak{m}}(u+\theta) = \frac{\Gamma(u)}{\Gamma(u+\theta)\Gamma(1-\theta)} W_\phi(u+\theta) = C\mathcal{M}_{B(1,\theta)}(u)\mathcal{M}_{m^\uparrow}(u),$$

where $C = \frac{W_\phi(1+\theta)}{\Gamma(1-\theta)\Gamma(1+\theta)}$ and $B(1,\theta)$ is a Beta distribution of parameter $(1,\theta)$. Hence by the formula for the density of product of random variables, we have, for x large enough such that $m^\uparrow$ is non-increasing on (x,∞),

$$\begin{aligned}
\frac{1}{C}\mathfrak{m}(x)p_\theta(x) &= \int_x^\infty m^\uparrow(y)\left(1-\frac{x}{y}\right)^{\theta-1}\frac{1}{y}dy = \int_x^\infty y^{-\theta} m^\uparrow(y)(y-x)^{\theta-1}dy \\
&\geq \int_x^{x+1} y^{-\theta} m^\uparrow(y)(y-x)^{\theta-1}dy \geq (x+1)^{-\theta} m^\uparrow(x+1) \geq C_\psi x^{-\theta} m^\uparrow(x)
\end{aligned}$$

for some $C_\psi > 0$ by [91, Theorem 5.5 (1)]. Combine the above relations together, we have, for x large enough,

$$\frac{m^\uparrow(x)}{x^{2\theta}\mathfrak{m}(x)} \leq \frac{1}{CC_\psi}.$$

Now denoting $m_n^\uparrow = \frac{\mathcal{R}^{(n)} m^\uparrow}{m^\uparrow}$, which is in $\mathrm{L}^2(m^\uparrow)$ by [91, Theorem 8.1], then we have $(\mathfrak{m}_n^\dagger(x))^2\mathfrak{m}(x) = (m_n^\uparrow(x))^2 m^\uparrow(x)\frac{m^\uparrow(x)}{x^{2\theta}\mathfrak{m}(x)} \leq \frac{1}{CC_\psi}(m_n^\uparrow(x))^2 m^\uparrow(x)$ and is integrable around ∞. Hence $\mathfrak{m}_n^\dagger \in \mathrm{L}^2(\mathfrak{m})$ for all $n \in \mathbb{N}$. Furthermore, again by [91, Theorem 8.1], $m_n^\uparrow$ is the co-eigenfunction for $P_t^\uparrow$ with eigenvalue e^{-nt}. Hence we have, for any $n \in \mathbb{N}$,

$$\begin{aligned}
\left\langle P_t^\dagger f, \mathfrak{m}_n^\dagger\right\rangle_{\mathfrak{m}} &= e^{-\theta t}\left\langle p_\theta P_t^\uparrow \frac{f}{p_\theta}, \frac{\mathcal{R}^{(n)} m^\uparrow}{p_\theta \mathfrak{m}}\right\rangle_{\mathfrak{m}} = e^{-\theta t}\left\langle P_t^\uparrow \frac{f}{p_\theta}, m_n^\uparrow\right\rangle_{m^\uparrow} \\
&= e^{-(n+\theta)t}\left\langle \frac{f}{p_\theta}, m_n^\uparrow\right\rangle_{m^\uparrow} = e^{-(n+\theta)t}\left\langle f, \frac{m_n^\uparrow m^\uparrow}{p_\theta \mathfrak{m}}\right\rangle_{\mathfrak{m}} = e^{-(n+\theta)t}\left\langle f, \mathfrak{m}_n^\dagger\right\rangle_{\mathfrak{m}}.
\end{aligned}$$

Therefore $\mathfrak{m}_n^\dagger$ is a co-eigenfunction for $P_t^\dagger$ with eigenvalue $e^{-(n+\theta)t}$. On the other hand, any solution f of the equation $\widehat{\Lambda}_\phi f = \mathcal{L}_n^\dagger$ shall satisfy the relation

$$\frac{\Gamma(1-\theta)}{W_\phi(1+\theta)} m(x)\mathcal{L}_n^\dagger(x) \overset{a.e.}{=} \int_0^\infty y^{-1} f(xy)\mathfrak{m}(xy)\iota\left(\frac{1}{y}\right)dy.$$

Hence taking Mellin transform on both sides and after some careful computations, we have

$$\mathcal{M}_{\mathfrak{m}f}(u) = \frac{(-1)^n}{n!}\frac{\Gamma(u-\theta)}{\Gamma(u-\theta-n)}\frac{W_\phi(u)}{W_\phi(1+\theta)} = \mathcal{M}_{w_n^\dagger}(u).$$

Therefore we see that $\mathfrak{m}_n^\dagger$ is a solution of $\widehat{\Lambda}_\phi f = \mathcal{L}_n^\dagger$ by injectivity of the Mellin transform, and the uniqueness of this solution is due to the one-to-one property of $\widehat{\Lambda}_\phi$. Hence the biorthogonality of $(\mathcal{P}_n^\dagger, \mathfrak{m}_n^\dagger)_{n\geq 0}$ follows by a similar argument as (3.97). This completes the proof.

Proof of Theorem 3.4.2(3)

First, take any $f \in \mathrm{Ran}(\Lambda_\phi)$ with $\Lambda_\phi g = f$ for some $g \in \mathrm{L}^2(m)$, then by the intertwining relation (3.43) and the spectral expansion for Q_t, see (3.70), we have

$$P_t f(x) = P_t\Lambda_\phi g(x) = \Lambda_\phi Q_t g(x) = \Lambda_\phi \sum_{n\geq 0} e^{-nt}\mathfrak{c}_n(-\theta)\langle g, \mathcal{L}_n\rangle_m \mathcal{L}_n(x) = \sum_{n\geq 0} e^{-nt}\langle g, \mathcal{L}_n\rangle_m \mathcal{P}_n(x),$$

where the last identity is justified by the fact that $\Lambda_\phi \in \mathbf{B}(\mathrm{L}^2(m), \mathrm{L}^2(\mathfrak{m}))$, the Bessel property of $\left(\mathfrak{c}_n^{-\frac{1}{2}}(-\theta)\mathcal{P}_n\right)_{n\geq 0}$ combined with the fact that the sequence $\left(\sqrt{\mathfrak{c}_n(-\theta)}e^{-nt}\langle g, \mathcal{L}_n\rangle_m\right)_{n\geq 0} \in \ell^2$ since $(\langle g, \mathcal{L}_n\rangle_m)_{n\geq 0} \in \ell^2$. Moreover, recalling that $\widehat{\Lambda}_\phi \mathfrak{m}_n = \mathcal{L}_n$, we see that $\langle g, \mathcal{L}_n\rangle_m = \left\langle \Lambda_\phi g, \mathfrak{m}_n\right\rangle_{\mathfrak{m}} = \langle f, \mathfrak{m}_n\rangle_{\mathfrak{m}}$, hence this proves (3.85) for all $(\psi, f) \in \mathcal{D}^{\checkmark}(\Lambda_\phi)$. Now let us define the spectral operator S_t, $t \geq 0$, by

$$S_t f(x) = \sum_{n=0}^{\infty} e^{-nt}\langle f, \mathfrak{m}_n\rangle_{\mathfrak{m}} \mathcal{P}_n(x). \tag{3.99}$$

We first note that under the condition $\mathcal{D}^{\mathcal{N}_P}(\mathfrak{m})$,

$$\sqrt{\mathfrak{c}_n(-\theta)}e^{-nt}\langle f, \mathfrak{m}_n\rangle_{\mathfrak{m}} \leq e^{-nt}\|f\|_{\mathfrak{m}}\|\mathfrak{m}_n\|_{\mathfrak{m}} = O\left(n^{\frac{\theta}{2}}e^{(-t+\epsilon)n}\right).$$

Hence $(\sqrt{\mathfrak{c}_n(-\theta)}e^{-nt}\langle f, \mathfrak{m}_n\rangle_{\mathfrak{m}})_{n\geq 0} \in \ell^2$. By the Bessel property of the sequence $\left(\mathfrak{c}_n^{-\frac{1}{2}}(-\theta)\mathcal{P}_n\right)_{n\geq 0}$, we get that $S_t f(x) \in \mathrm{L}^2(\mathfrak{m})$ for $(\psi, f) \in \mathcal{D}^{\checkmark}(\Lambda_\phi) \cup \mathcal{D}^{\mathcal{N}_P}(\mathfrak{m})$. Our next aim is to show $P_t f(x) = S_t f(x)$ under the conditions $\mathcal{D}^{\mathcal{N}_P}(\mathfrak{m})\backslash\mathcal{D}^{\checkmark}(\Lambda_\phi)$. Since $\mathrm{Ran}(\Lambda_\phi)$ is dense in $\mathrm{L}^2(\mathfrak{m})$, for any $f \in \mathrm{L}^2(\mathfrak{m})$, there exists a sequence $(g_m)_{m\geq 0} \in \mathrm{L}^2(m)$ such that $\lim_{m\to\infty} \Lambda_\phi g_m = f$ in $\mathrm{L}^2(\mathfrak{m})$. Hence we have from the previous part that

$$P_t \Lambda_\phi g_m(x) = \sum_{n=0}^{\infty} c_{n,t}(\Lambda_\phi g_m) \mathfrak{c}_n^{-\frac{1}{2}}(-\theta) \mathcal{P}_n(x),$$

where the constants $c_{n,t}$ are defined by $c_{n,t}(f) = \sqrt{\mathfrak{c}_n(-\theta)} e^{-nt} \langle f, \mathfrak{m}_n \rangle_{\mathfrak{m}}$ for $f \in \mathrm{L}^2(\mathfrak{m})$. Now let us define operator $\mathcal{S} : \ell^2 \to \mathrm{L}^2(\mathfrak{m})$ by, for any $(c_n)_{n\geq 0} \in \ell^2$,

$$\mathcal{S}((c_n)) = \sum_{n=0}^{\infty} c_n \mathfrak{c}_n^{-\frac{1}{2}}(-\theta) \mathcal{P}_n. \tag{3.100}$$

Then by [91, (2.5)], $\mathcal{S}$ is a bounded operator with operator norm $|||\mathcal{S}|||$ and

$$||P_t \Lambda_\phi g_m - S_t f||_{\mathfrak{m}}^2 = ||\mathcal{S}(c_{n,t}(\Lambda_\phi g_m - f))||_{\mathfrak{m}}^2 \leq |||\mathcal{S}||| \sum_{n=0}^{\infty} c_{n,t}^2(\Lambda_\phi g_m - f) \leq C_t ||\Lambda_\phi g_m - f||_{\mathfrak{m}}^2$$

for some constant $0 < C_t < \infty$. Hence $\lim_{m\to\infty} P_t \Lambda_\phi g_m = S_t f$. However, since P_t is a contraction, we conclude that $P_t f = S_t f$ under $\mathcal{D}^{\mathcal{N}_P}(\mathfrak{m})$. The spectral expansion of $P_t^\dagger f$ for $(\psi, f) \in \mathcal{D}^{\checkmark}(\Lambda_\phi)$ can be proved similarly using the spectral expansion of $Q_t^\dagger f$ in (3.71), the intertwining between $P^\dagger$ and $Q^\dagger$, and the properties of $\mathcal{P}_n^\dagger$ as well as $\mathfrak{m}_n^\dagger$. Finally, for $(\psi, f) \in \mathcal{D}^{\mathcal{N}_P}(\mathfrak{m})$, we have $\psi_\uparrow \in \mathcal{N}_P \cap \mathcal{N}_\uparrow$ and therefore by [91, Theorem 1.11], for all $f \in \mathrm{L}^2(m^\uparrow)$,

$$P_t^\uparrow f = \sum_{n=0}^{\infty} e^{-nt} \left\langle f, m_n^\uparrow \right\rangle_{m^\uparrow} \mathcal{P}_n^{\psi_\uparrow}.$$

Hence

$$P_t^\dagger f = e^{-\theta t} p_\theta P_t^\uparrow \left(\frac{f}{p_\theta} \right) = \sum_{n=0}^{\infty} e^{-(n+\theta)t} \left\langle \frac{f}{p_\theta}, m_n^\uparrow \right\rangle_{m^\uparrow} \mathcal{P}_n^\dagger = \sum_{n=0}^{\infty} e^{-(n+\theta)t} \left\langle f, \mathfrak{m}_n^\dagger \right\rangle_{\mathfrak{m}} \mathcal{P}_n^\dagger.$$

This completes the proof of Theorem 3.4.2.

Proof of Corollary 3.4.1

For any $\psi \in \mathcal{N}_P \cap \mathcal{N}_{\checkmark}$ and assuming $\overline{\overline{\Pi}}(0+) < \infty$, since by Theorem 3.4.2, $\left(\mathfrak{c}_n^{-\frac{1}{2}}(-\theta)\mathcal{P}_n\right)_{n\geq 0}$ and $(\sqrt{\mathfrak{c}_n(\mathfrak{b})}\mathfrak{m}_n)_{n\geq 0}$ are both Bessel sequences in $\mathrm{L}^2(\mathfrak{m})$ with bound 1, we have, for $t > T_{\mathfrak{b}} = \frac{1}{2}\ln\left(\frac{\mathfrak{b}+2}{2-\theta}\right)$,

$$\begin{aligned}
\|P_t f - \mathfrak{m}f\|_{\mathfrak{m}}^2 = \|\mathcal{S}(c_{n,t}(f))\|_{\mathfrak{m}}^2 &\leq \sum_{n=1}^{\infty} \frac{\mathfrak{c}_n(-\theta)}{\mathfrak{c}_n(\mathfrak{b})} \left|\left\langle P_t f, \sqrt{\mathfrak{c}_n(\mathfrak{b})}\mathfrak{m}_n \right\rangle_{\mathfrak{m}}\right|^2 \\
&= e^{-2t} \sum_{n=1}^{\infty} \frac{e^{-2(n-1)t}\mathfrak{c}_n(-\theta)}{\mathfrak{c}_n(\mathfrak{b})} \left|\left\langle f, \sqrt{\mathfrak{c}_n(\mathfrak{b})}\mathfrak{m}_n \right\rangle_{\mathfrak{m}}\right|^2 \\
&= \frac{e^{-2t}\mathfrak{c}_1(-\theta)}{\mathfrak{c}_1(\mathfrak{b})} \sum_{n=1}^{\infty} \frac{e^{-2(n-1)t}\mathfrak{c}_1(\mathfrak{b})\mathfrak{c}_n(-\theta)}{\mathfrak{c}_n(\mathfrak{b})\mathfrak{c}_1(-\theta)} \left|\left\langle f - \mathfrak{m}f, \sqrt{\mathfrak{c}_n(\mathfrak{b})}\mathfrak{m}_n \right\rangle_{\mathfrak{m}}\right|^2 \\
&\leq \frac{\mathfrak{b}+1}{1-\theta} e^{-2t} \sum_{n=1}^{\infty} \left|\left\langle f - \mathfrak{m}f, \sqrt{\mathfrak{c}_n(\mathfrak{b})}\mathfrak{m}_n \right\rangle_{\mathfrak{m}}\right|^2 \\
&\leq \frac{\mathfrak{b}+1}{1-\theta} e^{-2t} \|f - \mathfrak{m}f\|_{\mathfrak{m}}^2,
\end{aligned}$$

where we used the fact that by the Stirling approximation, $\frac{e^{-2(n-1)t}\mathfrak{c}_1(\mathfrak{b})\mathfrak{c}_n(-\theta)}{\mathfrak{c}_n(\mathfrak{b})\mathfrak{c}_1(-\theta)} \leq 1$ for all $t > T_{\mathfrak{b}}$. On the other hand, for $t \leq T_{\mathfrak{b}}$, $\frac{\mathfrak{b}+1}{1-\theta}e^{-2t} \geq \frac{\mathfrak{b}+1}{\mathfrak{b}+2}\frac{2-\theta}{1-\theta} \geq 1$ since $\mathfrak{b} \geq 0 > -\theta$. Invoking that P_t is a contraction, this concludes the proof of this corollary.

CHAPTER 4

RISK-NEUTRALIZATION TECHNIQUES AND EXAMPLES

4.1 Introduction

When using a stochastic process to model a financial asset and perform derivative pricing, one shall always refer to the Fundamental Theorem of Asset Pricing (FTAP), which was first suggested by M. Harrison and D. Kreps in 1979, and was generalized by M. Harrison and S. Pliska (1981) as well as F. Dalbaen and W. Schachermayer (1994). The notion of arbitrage is crucial in the modern theory of finance. An *arbitrage opportunity* is the possibility to make a profit in a financial market without risk and without net investment of capital. The principle of *no arbitrage* states that a mathematical model of a financial market should not allow for arbitrage possibilities. The basic message of the FTAP is that a model of a financial market is free of arbitrage if and only if there is a probability measure $\mathbb{Q}$, equivalent to the original (real-world, physical) probability measure $\mathbb{P}$ such that any price process is a martingale under $\mathbb{Q}$. This theorem was proved by M. Harrison and S. Pliska [57] in 1981 for the case where the underlying probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ is finite. In the same year D. Kreps [64] extended this theorem to a more general setting: A bounded process $X = (X_t)_{0\leq t\leq T}$ admits no free lunch if and only if there is a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ such that X is a martingale under $\mathbb{Q}$. Delbaen and Schachermayer [39] state the most general case of FTAP as follows, where we also refer to [39] for the definition of the concepts of sigma/semi-martingales, and No Free Lunch with Vanishing Risk which is a mild strengthening of the concept of No Arbitrage.

Theorem 4.1.1. *A semi-martingale $X = (X_t)_{0\leq t\leq T}$ admits no free lunch with vanishing* *risk if and only if there is a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ such that X is a sigma-martingale under $\mathbb{Q}$.*

If X is bounded (resp. locally bounded) the term sigma-martingale may equivalently be replaced by the term martingale (resp. local martingale).

In this case, the price of a derivative is indeed the discounted expected value of the future payoff under the risk-neutral measure. This risk-neutral probability measure generally differs from its statistical (real-world or physical) counterpart. The latter one describes the likelihood of these risky outcomes and is typically estimated from historical time series data on past realizations. The risk neutral probability, on the other hand, is the market price of Arrow-Debreu securities associated with risky events. The question then arises as to how one may construct the risk neutral density from the estimated statistical density for such risks, to price contingent claims written on these uncertainties.

The traditional way of risk-neutral valuation is to do a change of measure using Girsanov's theorem. Girsanov's theorem describes how the dynamics of stochastic processes change when the original measure is changed to an equivalent probability measure. In mathematical finance, this theorem will tell us how to convert from the physical measure to the risk-neutral measure. However, one of the drawbacks of this method is that it can only be used in a limited number of situations, namely when the pricing process is modeled by a diffusion adapted to the natural filtration of the Wiener process. When the process is not a diffusion, it is very hard to apply the Girsanov way of risk-neutralization. Moreover, even when one starts with a stochastic process which has various properties that can capture the behavior of a financial asset, and are easily tractable, after changing the measure although it becomes risk-neutral but in most of the cases it loses its tractability.

On the other hand, there are some additional issues in risk-neutralization. For example, one major issue is that even if we are already under a risk-neutral measure, we may need to model some sectors that are not traded in the market, which do not have to follow the local-martingale requirement due to their non-tradability. Then the question naturally arises that, is it possible to represent the traded asset as a function or transformation of these non-traded sectors, such that the discounted transformed process is a local martingale under the same measure? For example, we may want to represent a stock index in terms of macro-economic data, or represent a firm's stock price in terms of its (non-traded) asset value, etc. This is particularly useful when we consider Merton's structural model of credit risk, as suggested by Merton in 1974 [76], which has long been criticized for being unrealistic because a firm's value is not tradable.

To overcome this, we suggest several transformations on a tractable process (or equivalently, its respective semigroup) in order to make the discounted transformed process a martingale, while still keeping its tractability. We refer to such procedures as *risk-neutralization* transformations. In particular, we suggest using an intertwining relation between semigroups to achieve this goal. Moreover, we provide examples that illustrate several classes of processes (Lévy, self-similar and generalized CIR) to reveal the usefulness of our result, and furthermore, show that under certain circumstances, the derivative pricing formula can be represented by spectral expansions and evaluated numerically.

The paper is organized as follows. In section 1, we describe the risk-neutralization method and some corollaries with their proofs. In section 2, we apply above mentioned methods on some important classes of Markov processes (Lévy processes, Self-similar processes and Generalized CIR (Laguerre) processes), and some special instances of those processes. In section 3, we show why these methods are useful and numerically tractable when one does pricing of derivatives.

4.1.1 Preliminaries

Let $X = (X_t)_{t\geq 0}$ be a Markov process on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ and state space $E \subseteq \mathbb{R}$, and denote its semigroup by $P = (P_t)_{t\geq 0}$, i.e. for all $x \in E,\ t \geq 0$,

$$P_t f(x) = \mathbb{E}_x[f(X_t)],$$

for $f \in \mathcal{B}(E)$ such that $P_t|f| < \infty$, where $\mathbb{E}_x$ denotes the expectation associated to $\mathbb{P}_x(X_0 = x) = 1$, and $\mathcal{B}(E)$ is the set of all measurable functions on E. We also define $\mathcal{B}_b(E)$, the set of all measurable bounded functions, and the shift operator $\theta = (\theta_t)_{t\geq 0}$, $\theta_t : \Omega \to \Omega$, with the property $X_s \circ \theta_t = X_{t+s}$, $t, s \geq 0$.

The infinitesimal generator of X (or of its semigroup P) is defined as

$$\mathbf{A}f = \lim_{t\to 0} \frac{P_t f - f}{t} \tag{4.1}$$

for $f \in \mathcal{D}(\mathbf{A}) = \{f \in \mathcal{B}_b(E);\ \lim_{t\to 0} \frac{P_t f - f}{t} \text{ exists in } \mathcal{B}_b(E)\}$. We call $\mathcal{D}(\mathbf{A})$ *the domain of the generator* $\mathbf{A}$. Furthermore, according to Dynkin's theorem, if $f \in \mathcal{D}(\mathbf{A})$, then the process

$$\left\{ f(X_t) - f(X_0) - \int_0^t \mathbf{A}f(X_s)ds \right\}_{t\geq 0} \tag{4.2}$$

is a martingale, see e.g. [108].

In practice, the payoff functions in financial modeling are not necessarily bounded. However, one can also define a version of the generator that acts on unbounded functions as follows, see e.g. [108].

Definition 4.1.1.1. *A function $f \in \mathcal{B}(E)$ is said to belong to* the domain $\mathcal{D}(\mathcal{A})$ of the full generator *if there exists a function $g \in \mathcal{B}(E)$ such that the function $t \to g(X_t)$ is integrable $\mathbb{P}_x$-a.s. for each $x \in E$ and the process*

$$\left\{ M_t^f := f(X_t) - f(X_0) - \int_0^t g(X_s)ds \right\}_{t \geq 0} \tag{4.3}$$

is a martingale. Then we write $g = \mathcal{A}f$ and $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ represents the full generator *of the process $(X_t)_{t\geq 0}$.*

Note that g is not uniquely defined. We identify all functions g such that (4.3) is a martingale and write $\mathcal{A}f$ instead of g. It is easy to note that the domain of the infinitesimal generator is contained in the domain of the full generator, i.e. $\mathcal{D}(\mathbf{A}) \subset \mathcal{D}(\mathcal{A})$. Moreover, we can equivalently define the full generator using the operator resolvent $(U_q)_{q>0}$ associated to the transition semigroup $P = (P_t)_{t\geq 0}$ by the following proposition.

Proposition 4.1.1. *Let $f \in \mathcal{B}(E)$ and $\lim_{t\to\infty} e^{-qt} P_t f(x) = 0$ for any $x \in E$. Then $f \in \mathcal{D}(\mathcal{A})$ if and only if there exists a function $g \in \mathcal{B}(E)$ such that the function $t \to g(X_t)$ is integrable $\mathbb{P}_x$-a.s, $U_q|g| < \infty$, and*

$$U_q(qf - g) = f \tag{4.4}$$

for all $q > 0$, where $U_q f(x) = \int_0^\infty e^{-qt} P_t f(x) dt$ is the q-resolvent of P. The function $g = \mathcal{A}f$ is uniquely determined up to a set of potential zero, that is, a set $C \subseteq E$ such that $U_q \mathbf{1}_C = 0$ for all $q > 0$.

Proof. First, let $f \in \mathcal{D}(\mathcal{A})$. Then, by the definition of the full generator, there exists a function $g \in \mathcal{B}(E)$ such that M^f defined by (4.3) is a martingale. Therefore, for any $t \geq 0$ and $x \in E$, $\mathbb{E}_x[M_t^f] = 0$, or, equivalently,

$$\mathbb{E}_x\left[f(X_t) - f(X_0) - \int_0^t g(X_s)ds\right] = P_t f(x) - f(x) - \int_0^t P_s g(x)ds = 0. \tag{4.5}$$

Then, for all $q > 0$, we have

$$\begin{aligned}
U_q(qf - g)(x) &= q\int_0^\infty e^{-qt} P_t f(x)dt - \int_0^\infty e^{-qt} P_t g(x)dt \\
&= q\int_0^\infty e^{-qt}\left[f(x) + \int_0^t P_s g(x)ds\right]dt - \int_0^\infty e^{-qt} P_t g(x)dt \\
&= f(x) + q\int_0^\infty e^{-qt}\int_0^t P_s g(x)dsdt - \int_0^\infty e^{-qt} P_t g(x)dt \\
&= f(x) - e^{-qt}\int_0^t P_s g(x)ds\Big|_0^\infty = f(x) + \lim_{t\to\infty}\left[e^{-qt}(P_t f(x) - f(x))\right] = f(x),
\end{aligned}$$

where we used that by Lebesgue's theorem $\frac{d}{dt}\int_0^t P_s g(x)dx = P_t g(x)$, and $\lim_{t\to\infty} e^{-qt} P_t f(x)dt = 0$ for any $x \in E$, hence (4.4) holds. Conversely, assume there exists a function g such that (4.4) is true. Denote

$$l_g(x) := \lim_{t\to\infty} e^{-qt}\int_0^t P_s g(x)ds, \tag{4.6}$$

and note that for any $x \in E$,

$$|l_g(x)| = \left|\lim_{t\to\infty} e^{-qt}\int_0^t P_s g(x)ds\right| \leq \int_0^\infty e^{-qs} P_s|g|(x)ds = qU_q|g|(x) < \infty.$$

Therefore for any $x \in E$, $l_g(x)$ is finite, and we will actually show that $l_g \equiv 0$. Then, by integration by parts, we have

$$U_q g(x) = q\int_0^\infty e^{-qt}\int_0^t P_s g(x)dsdt + l_g(x) = q\int_0^\infty e^{-qt}\left(\int_0^t P_s g(x)ds + l_g(x)\right)dt. \tag{4.7}$$

On the other hand, from (4.4), we get

$$U_q g(x) = qU_q f(x) - f(x) = q\int_0^\infty e^{-qt}\left(P_t f(x) - f(x)\right)dt. \tag{4.8}$$

Combining (4.7) and (4.8), and applying inverse Laplace transform, we see that for any $t \geq 0$ and $x \in E$,

$$P_t f(x) - f(x) - \int_0^t P_s g(x)ds - l_g(x) = 0. \tag{4.9}$$

Note that from this equation it follows that for any $x \in E$,

$$l_g(x) + \lim_{t\to\infty} e^{-qt} l_g(x) = \lim_{t\to\infty} e^{-qt} P_t f(x) - \lim_{t\to\infty} e^{-qt} f(x) = 0, \tag{4.10}$$

which means that $l_g \equiv 0$. Therefore, for any $t \geq 0$,

$$P_t f(x) - f(x) - \int_0^t P_s g(x) ds = 0. \tag{4.11}$$

Now, take any $u, t \geq 0$, then

$$\begin{aligned}
\mathbb{E}_x\left[f(X_{t+u}) - f(X_0) - \int_0^{t+u} g(X_s)ds \middle| \mathcal{F}_u \right] &= \mathbb{E}_x\left[f(X_{t+u}) - \int_0^{t+u} g(X_s)ds \middle| \mathcal{F}_u \right] - f(X_0) \\
&= \mathbb{E}_x\left[\left(f(X_t) - \int_0^t g(X_s)ds \right) \circ \theta_u \middle| \mathcal{F}_u \right] - \int_0^u g(X_s)ds - f(X_0) \\
&= P_t f(X_u) - \int_0^t P_s g(X_u)ds - \int_0^u g(X_s)ds - f(X_0) \\
&= f(X_u) - f(X_0) - \int_0^u g(X_s)ds,
\end{aligned}$$

where in the last equality we used (4.11) for X_u. This implies that M^f defined by 4.3 is a martingale, and hence $f \in \mathcal{D}(\mathcal{A})$ and $g = \mathcal{A}f$. □

We will also need some basic concepts from potential theory, and we follow Dellacherie, Meyer [40, Chapter XII], for the definitions of the sets of excessive and invariant functions.

Definition 4.1.1.2. *Let $r \geq 0$.*

1. *The set of r-excessive functions for the semigroup P is defined as*

$$\mathcal{E}_r(P) = \{h_r : E \to \mathbb{R}_+; e^{-rt} P_t h_r(x) \leq h_r(x), \lim_{t\searrow 0} e^{-rt} P_t h_r(x) = h_r(x)\}.$$

   Moreover, h_r is called r-purely excessive if $h_r \in \mathcal{E}_r(P)$ and $\lim_{t\to\infty} e^{-rt} \mathcal{P}_t h_r(x) = 0$.

2. *The set of r-invariant functions for P is defined as*

$$\mathcal{I}_r(P) = \{h_r \in \mathcal{E}_r; e^{-rt} P_t h_r(x) = h_r(x)\}.$$

When $r = 0$, we will simply call them excessive or invariant functions.

Next, we have the following lemma that links terminologies from stochastic calculus to potential theory.

Lemma 4.1.1. *Let $r \geq 0$.*

1. *$h_r \in \mathcal{I}_r(P)$ if and only if $(e^{-rt}h_r(X_t))_{t\geq 0}$ is a positive martingale under $\mathbb{P}$.*
2. *$h_r \in \mathcal{E}_r(P)$ if and only if $(e^{-rt}h_r(X_t))_{t\geq 0}$ is a positive super-martingale under $\mathbb{P}$.*
3. *If $(e^{-rt}h_r(X_t))_{t\geq 0}$ is a local martingale bounded from below under $\mathbb{P}$, then $h_r \in \mathcal{E}_r(P)$.*

Therefore, it is natural for us to define the following set:

$$\mathcal{E}_r^{loc}(P) = \{h_r \in \mathcal{E}_r; (e^{-rt}h_r(X_t))_{t\geq 0} \text{ is a local martingale}\}. \tag{4.12}$$

Proof. First, let $h_r \in \mathcal{I}_r(P)$, then clearly $P_t|h_r|(x) = P_t h_r(x) = e^r t h_r(x) < \infty$ for any $t \geq 0$ and $x \in E$. Moreover, for any $s < t$, the Markov property entails that

$$\mathbb{E}\left[e^{-rt}h_r(X_t)|\mathcal{F}_s\right] = e^{-rt}\mathbb{E}[h_r(X_{t-s}) \circ \theta_s|\mathcal{F}_s] = e^{-rt}P_{t-s}h_r(X_s) = e^{-rs}h_r(X_s),$$

where for the last identity we use the fact that $h_r \in \mathcal{I}_r(P)$. Hence, $(e^{-rt}h_r(X_t))_{t\geq 0}$ is a positive martingale under $\mathbb{P}$. The reverse statement is obvious. Item (2) is proved similarly. Finally, to show item (3), we recall that every local martingale which is bounded from below is a super-martingale. Therefore, $(e^{-rt}h_r(X_t))_{t\geq 0}$ is a super-martingale under $\mathbb{P}$, and by item (2), $h_r \in \mathcal{E}_r(P)$. □

The requirement of discounted prices being martingales can be reinterpreted from the viewpoint of semigroups, and the concept of the *pricing semigroups* in financial economics goes back to Garman (1985) [52]. More precisely, let the stock price have dynamics $X = (X_t)_{t\geq 0}$, and recall that we call a measure $\mathbb{P}$ a *risk-neutral* measure, if $(e^{-rt}X_t)_{t\geq 0}$ is a martingale under $\mathbb{P}$, where $r \geq 0$ is the interest rate. The collection of operators $P^{(r)} = (e^{-rt}P_t)_{t\geq 0}$ defined by $P_t^{(r)} f(x) := \mathbb{E}_x[e^{-rt} f(X_t)]$ is referred as the *pricing semigroup* of S. We will then call $P = (P_t)_{t\geq 0}$ an *r-pricing semigroup*. Now we present a potential-theoretical characterization of pricing semigroups as follows.

Proposition 4.1.2. *For any $r \geq 0$, P is an r-pricing semigroup if $p_1 \in \mathcal{I}_r(P)$, where $p_1(x) = x$ is the identity function.*

Remark 4.1.1. *Note that by* (4.4)*, this condition is equivalent to $p_1 = qU_q^{(r)} p_1 = qU_{q+r} p_1$ for all $q > 0$, where $(U_q^{(r)})_{q>0}$ is the q-resolvent for $P^{(r)}$.*

Proof. Let $p_1 \in \mathcal{I}_r(P)$. Then, by part (1) of Lemma 4.1.1, $(e^{-rt}X_t)$ is a martingale under the measure $\mathbb{P}$. Therefore, $(e^{-rt}P_t)_{t\geq 0}$ defines a pricing semigroup, and P is an r-pricing semigroup. $\square$

Now, having presented some required definitions and notations, we are ready to describe our risk-neutralization techniques in the following section.

4.2 Risk-neutralization transformations

As we have already mentioned, to perform derivatives pricing, one should always refer to the Fundamental Theorem of Asset Pricing. Therefore, to price derivatives using the risk-neutral pricing approach, one has to identify a risk-neutral measure under which the discounted price process is a (local) martingale. To do this, we suggest a transformations on a tractable and flexible process (or equivalently, its respective semigroup), based on a concept of intertwining relationships, in order to make the discounted transformed process a martingale, while still keeping its tractability. Recall that we refer to such procedures as *risk-neutralization* transformations.

4.2.1 An intertwining approach

We are now ready to state our first risk-neutralization technique, which involves an intertwining relationship between semigroups. We propose the concept of intertwining relation between Markov semigroups as a comprehensive tool to develop some risk-neutralization techniques. We emphasize that the literature on intertwining is important with a borad range of applications in stochastic and functional analysis (see e.g. Dynkin [47], Rogers and Pitman [103], Diaconis and Fill [42], Patie and Savov [91]).

Theorem 4.2.1. *Let $P = (P_t)_{t\geq 0}$ and $Q = (Q_t)_{t\geq 0}$ be two semigroups, and assume that there exists a linear operator Λ such that for any $t \geq 0$*

$$P_t \Lambda p_1 = \Lambda Q_t p_1. \tag{4.13}$$

Then, Q is an r-pricing semigroup if one of the following conditions holds.

- *a) $\Lambda p_1 \in \mathcal{I}_r(P)$ and Λ is injective.*
- *b1) P is an r-pricing semigroup and b2) $\exists c > 0$ s.t. $\Lambda p_1 = c p_1$.*

Remark 4.2.1. *In the literature usually the intertwining relation* (4.13) *between two operators P and Q is given for all bounded measurable functions, i.e. $P_t \Lambda f = \Lambda Q_t f$ for all $t \geq 0$ and $f \in \mathcal{B}_b(E)$. However, we only require the identity* (4.13) *to hold for the function p_1, to make the theorem as comprehensive as possible.*

Proof. First, let us assume that *a*) holds. Since $\Lambda p_1 \in \mathcal{I}_r(P)$, then for any $t \geq 0$ one has

$$\Lambda Q_t p_1 = P_t \Lambda p_1 = e^{rt} \Lambda p_1.$$

Therefore, since we assumed that Λ is injective, $p_1 \in \mathcal{I}_r(Q)$, and Q is an r-pricing semigroup by Proposition 4.1.2.
Next, assume b) holds. Then, successfully using b2), (4.13), b1), the linearity of Λ and b2) again, for any $t \geq 0$ we get

$$Q_t p_1 = \frac{1}{c} Q_t \Lambda p_1 = \frac{1}{c} \Lambda P_t p_1 = \frac{1}{c} \Lambda e^{rt} p_1 = e^{rt} p_1.$$

Hence, $p_1 \in \mathcal{I}_r(Q)$, and according to Proposition 4.1.2, Q is an r-pricing semigroup, which concludes the proof. □

Remark 4.2.2. *Let $P = (P_t)_{t \geq 0}$ and $Q = (Q_t)_{t \geq 0}$ be the semigroups defined in Proposition 4.2.1, and denote the associated full generators by $\mathcal{A}$ and $\mathcal{G}$ respectively. Then, to find the pricing semigroups one can formulate Proposition 4.2.1, by Proposition 4.1.1, using an intertwining relation between the associated full generators, i.e.*

$$\mathcal{A} \Lambda f = \Lambda \mathcal{G} f, \tag{4.14}$$

for $f \in \mathcal{D}(\mathcal{G})$ s.t. $\Lambda f \in \mathcal{D}(\mathcal{A})$. It is also important to note that if one wants to consider local martingales instead of true martingales, then Lemma 4.1.1 tells us to focus on excessive functions instead of their invariant counterparts. In that case, to get similar results it is more convenient to work with extended *generators, see e.g. Bujorianu [26] for the definition of extended generators.*

Here it is useful to recall that if functions f and g are r-excessive for some $r \geq 0$, and $f = g$ except of a potential zero set then $f = g$ everywhere, see e.g. Blumenthal et al. [21, P. 80]. Moreover, since the full generator is also uniquely determined up to the zero potential set by equation (4.4), *we notice that for our purposes there is a* *one-to-one correspondence between invariant (resp. excessive) functions and their full (resp. extended) generators.*

We proceed by providing two instances of this risk-neutralization technique using an intertwining relation that correspond to some known transformations and can be easily identified. The first one hinges on an observation from Dynkin [47], and the second one is related to Doob's h-transform and the so-called eigenmeasure defined in [98].

Corollary 4.2.1. *Assume there exists a function $h_r \in \mathcal{I}_r(P)$ which is a homeomorphism. Then the the family of operators $Q = (Q_t)_{t\geq 0}$ defined by*

$$Q_t f(y) = P_t(f \circ h_r)(x), \quad y = h_r(x), \tag{4.15}$$

for measurable functions $f : E \to \mathbb{R}_+$, is an r-pricing semigroup.

Proof. First, note that since h_r is a homeomorphism, Q as defined by (4.15) is a Markov semigroup by Dynkin's criterion, see e.g. Carmona et al. [32]. Then, it is easy to check that the following intertwining relation

$$P_t \Lambda f = \Lambda Q_t f$$

with $\Lambda f(x) = f \circ h_r(x)$, holds. Therefore, since h_r is a homeomorphism, its inverse, h_r^{-1}, exists, and by defining $\Lambda^{-1} f = f \circ h_r^{-1}$, we note that it is the left inverse of Λ. Recalling that a linear operator has a left inverse if and only if it is injective, and noting that $\Lambda p_1 = h_r$, we see that condition a) of Theorem 4.2.1 holds. Hence, Q is an r-pricing semigroup. □

Corollary 4.2.2. *Let $H_\lambda : E \to \mathbb{R}_+$ be a strictly positive function such that $e^{-\lambda t} P_t H_\lambda(x) = H_\lambda(x)$ for some $\lambda \in \mathbb{R}$. Moreover, let $\mathfrak{s} := \frac{h_r}{H_\lambda}$ be a homeomorphism* *for some* $h_r \in \mathcal{I}_r(P)$, $r \geq \lambda$. *Then the family of operators* $Q = (Q_t)_{t\geq 0}$ *satisfying the intertwining relation*

$$P_t \Lambda f = \Lambda Q_t f \tag{4.16}$$

with $\Lambda f = (f \circ \mathfrak{s}) H_\lambda$, *is an r-pricing semigroup. Moreover, when* $r \geq \lambda$, *then* $Q^{(\lambda)} = \left(e^{-\lambda t} Q_t\right)_{t\geq 0}$ *is an* $(r-\lambda)$*-pricing semigroup.*

Proof. First, note that through identity (4.16) we can be equivalently define the operator Q as follows:

$$Q_t f(y) = \frac{1}{H_\lambda(x)} P_t(H_\lambda(f \circ \mathfrak{s}))(x), \quad y = \mathfrak{s}(x). \tag{4.17}$$

Therefore Doob's h-transform and the assumption of $\mathfrak{s}$ being a homeomorphism insure that Q indeed is a Markov semigroup again by Dynkin's criterion. Now, on one hand, we have

$$\Lambda p_1 = (p_1 \circ \mathfrak{s}) H_\lambda = \mathfrak{s} H_\lambda = h_r.$$

On the other hand since $h_r \in \mathcal{I}_r(P)$, by an application of Proposition 4.2.1, Q is an r-pricing semigroup. Moreover, it is easy to see that since $h_r \in \mathcal{I}_r(P)$, then when $r \geq \lambda$, h_r is an $(r-\lambda)$-invariant function for the semigroup $P^{(\lambda)} = (e^{-\lambda t} P_t)_{t\geq 0}$. Therefore, $Q^{(\lambda)} = (e^{-\lambda t} Q_t)_{t\geq 0}$ is an $(r-\lambda)$-pricing semigroup, and this concludes the proof. □

In Remark 4.3.1 we will show that Corollary 4.2.2 is a generalization of the well-known Esscher transform. We would also like to point out how Corollary 4.2.2 can be useful when considering the famous Ross recovery theorem, see Ross [104]. The objective of the Ross recovery theorem is to find the physical (real-world) probability measure, $\mathbb{P}$, under Markovian structure assuming that the risk-neutral measure, $\mathbb{Q}$, and the short interest rate function are known ex ante.

Later on, many authors extended the original Ross recovery theorem, see e.g. Carr, Yu [29], Dubinskiy and Goldstein[44], Qin, Linetsky [97], Walden [115], Borovicka et al. [24]. In particular, Park [84] describes all possible beliefs of market participants on physical measures under Markovian environments when a risk-neutral measure is given, using the Martin integral representation of Markovian pricing kernels (or numeraires).

More precisely, Ross [104] assumed that all prices depend only on a single driver $X = (X_t)_{t\geq 0}$, and the short interest rate r_t is determined by X_t, i.e. there is a continuous function $r(\cdot)$ such that $r_t = r(X_t)$. Further, it is assumed that the pricing kernel is Markovian. Define a pricing operator $P = (P_t)_{t\geq 0}$ by

$$P_t f(x) := \mathbb{E}\left[e^{-\int_0^t r(X_s)ds} f(X_t)\right], \tag{4.18}$$

and denote the corresponding infinitesimal generator by $\mathcal{L}$. Then, Ross recovery theorem argues that if there exists a positive solution h of $\mathcal{L}h = -\lambda h$ such that

$$\left(e^{\lambda t-\int_0^t r(X_s)ds}\frac{h(X_t)}{h(X_0)}\right)_{t\geq 0}$$

is a martingale under $\mathbb{Q}$, than such a pair (h, λ) is unique, X_t is recurrent under the corresponding transformed measure $\mathbb{P}$, and hence Ross recovery is possible by

$$\left.\frac{d\mathbb{P}}{d\mathbb{Q}}\right|_{\mathcal{F}_t} = e^{\lambda t-\int_0^t r(X_s)ds}\frac{h(X_t)}{h(X_0)}. \tag{4.19}$$

Therefore, to recover the physical probability measure $\mathbb{P}$, one can first construct its risk-neutral counterpart $\mathbb{Q}$ using Corollary 4.2.2, and then get the physical one through equation (4.19).

4.3 Examples

Now that we presented the risk-neutralization techniques, we are ready to illustrate how these transformations can be applied on some classical processes which are used in finance because of their flexibility and tractability, in particular for doing risk-neutral pricing of derivatives. Namely, we will focus on Lévy processes (with two-sided jumps), self-similar processes, and generalized CIR (Laguerre) models. Moreover, we provide analytical formulas which are very tractable from the computational point of view, and which can also be used for numerical estimation of model parameters.

4.3.1 Lévy processes

Let $\xi = (\xi_t)_{t\geq 0}$ be a real-valued Lévy process, that is a real-valued random process with stationary and independent increments, defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ with characteristic triplet (σ, m, Π), where $\sigma \geq 0$, $m \in \mathbb{R}$, and Π is the Lévy measure concentrated on $\mathbb{R} \setminus \{0\}$ and satisfying $\int_{\mathbb{R}}(1 \wedge x^2)\Pi(dx) < \infty$. Let $X = (e^{\xi_t})_{t\geq 0}$, and denote its semigroup by $P = (P_t)_{t\geq 0}$, i.e. $P_t f(x) = \mathbb{E}_x[f(X_t)] = \mathbb{E}_{\ln x}[f(e^{\xi_t})]$ for any $t \geq 0$ and $x \in \mathbb{R}$. Then a well-known fact, see e.g. Kyprianou [67], shows that the law of ξ is characterized by the characteristic exponent Ψ, such that for any $t \geq 0$ and $x \in \mathbb{R}$, $P_t p_z(x) = e^{t\Psi(z)} p_z(x)$, where $p_z(x) = x^z$, and Ψ is in the form

$$\Psi(z) = \sigma^2 z^2 + mz + \int_{-\infty}^{\infty} (e^{zx} - 1 - zx1_{\{|x|<1\}})\Pi(dx).$$

Now let

$$C := \left\{u \in \mathbb{R}; \int_{|x|>1} e^{ux}\Pi(dx) < \infty\right\} \neq \emptyset,$$

then Ψ admits an analytical extension on $\Re(z) \in C$ and $\Psi''(u) > 0$ in the interior of C by Sato [106, Lemma 26.4]. Therefore, $\Psi : C \to \mathbb{R}$ is a convex function on C. Let θ denote the largest root of Ψ, which is either 0 (when $\Psi'(0+) \geq 0$), or strictly greater than 0 (when $\Psi'(0+) < 0$). Let M denote the supremum of C which can be infinity, and note that Ψ is increasing on $[\theta, M)$, hence there exists a function $\Phi : [0, \Psi(M_-)) \to [\theta, M)$ which is the inverse of Ψ, where $\Psi(M_-) = \lim_{u \uparrow M} \Psi(u)$ and can be either finite or ∞.

Proposition 4.3.1. *For any $0 < r \leq \Psi(M_-)$, define the family of operators $Q = (Q_t)_{t\geq 0}$ by the following intertwining relation*

$$P_t \Lambda f = \Lambda Q_t f,$$

where $\Lambda f(x) = f\left(x^{\Phi(r)}\right)$. Then, Q is an r-pricing semigroup.

Proof. Since $\int_1^\infty e^{ux}\Pi(dx) < \infty$ for all $u \in [0, M)$, by Sato [106, Theorem 25.17], we have

$$\mathbb{E}[e^{u\xi_t}] = e^{t\Psi(u)}$$

for all $u \in [0, M)$. Hence, defining $h_r(x) = x^{\Phi(r)}$, we have

$$P_t h_r(x) = \mathbb{E}_x[h_r(X_t)] = \mathbb{E}_x[X_t^{\Phi(r)}] = \mathbb{E}_{\ln x}[e^{\Phi(r)\xi_t}] = \mathbb{E}[e^{\Phi(r)(\xi_t + \ln x)}] = x^{\Phi(r)} e^{\Psi(\Phi(r))t} = h_r(x)e^{rt}.$$

Therefore, $h_r \in \mathcal{I}_r(P)$, and since h_r is a monotonic function, it is a homeomorphism. Hence, noting that we can write $\Lambda f(x) = f \circ h_r(x)$, Q is an r-pricing semigroup by direct application of Corollary 4.2.1. □

Remark 4.3.1. *In this remark we show that Corollary 4.2.2 is a generalization of the well-known Esscher transform. Let $(X_t = e^{\xi_t})_{t\geq 0}$, where $(\xi_t)_{t\geq 0}$ is a real-valued Lévy process with characterisitc exponent Ψ (see Section 4.3.1). Then, $P_t f(x) = \mathbb{E}_x[f(X_t)] = \mathbb{E}_{\ln x}[f(e^{\xi_t})]$. Further assume that for a fixed $r > 0$, there exists $u \in \mathbb{R}$ s.t. $u, u+1 \in C$,*

$z \mapsto \Psi(z)$ admits analytical extension on $\Re(z) \in (\min(u, u+1), \max(u, u+1))$ and $\Psi(u+1) - \Psi(u) = r$, and define $P^{(\Psi(u))} = (e^{-\Psi(u)t}P_t)_{t\geq 0}$. Next, it can be easily checked that $H_{\Psi(u)}(x) := x^u$ solves $e^{-\Psi(u)t}P_t H_{\Psi(u)}(x) = H_{\Psi(u)}(x)$, and $p_1 H_{\Psi(u)} \in \mathcal{I}_{\Psi(u+1)}(P)$, i.e. $\mathfrak{s} = p_1$ in Corollary 4.2.2. Therefore, if the semigroup $Q = (Q_t)_{t\geq 0}$ is defined via the following intertwining relation

$$P_t^{(\Psi(u))}\Lambda f(x) = \Lambda Q_t f(x), \tag{4.20}$$

where $\Lambda f(x) = (f \circ \mathfrak{s})H_{\Psi(u)}(x) = fH_{\Psi(u)}(x) = f(x)x^u$, then Q is an $\Psi(u+1) - \Psi(u) = r$-pricing semigroup by Corollary 4.2.2. Next, we observe that (4.20) *indeed can be written as another classical transformation, known as Doob's h transform, as follows:*

$$Q_t f(x) = \frac{1}{H_{\Psi(u)}} P_t^{(\Psi(u))}(H_{\Psi(u)}f)(x).$$

Recalling that h-transforms can be seen as a change of measure, we can equivalently say that $(e^{-rt}X_t)_{t\geq 0}$ is a martingale under the measure $\mathbb{Q}$ defined by

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = H_{\Psi(u)}(X_t)e^{-\Psi(u)t} = e^{u\xi_t - \Psi(u)t}.$$

In stock price modeling, this result is a classical transformation known as Esscher transform, *and it is widely used in options pricing.*

4.3.2 Self-Similar Processes

Self-similar processes are stochastic processes that are invariant in distribution under suitable scaling of time and space. These processes can typically be used to model random phenomena with long-range dependence. They are also increasingly important in many other fields of application, as there are economics and finance.

Recall that a *positive self-similar Markov process* $X = (X_t)_{t\geq 0}$ defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ is a cádlág Markov process which fulfills a scaling property, that is, there exists a constant $\alpha > 0$ such that for any $c > 0$, $(X_{ct})_{t\geq 0} \stackrel{d}{=} (c^\alpha X_t)_{t\geq 0}$ in the sense of equality of finite-dimensional distributions. If we let $K = (K_t)_{t\geq 0}$ be the semigroup associated with X, then the definition in terms of these operators will be equivalent to the following relation:

$$K_t f(cx) = K_{c^{-\frac{1}{\alpha}}t}(d_c f)(x),$$

where d_c is the dilation operator, i.e. $d_c f(x) = f(cx)$ for all $t, x, c > 0$.

Properties of positive self-similar Markov processes have been deeply studied by the early 1960s, especially through Lamperti's work on one-dimensional branching processes. Lamperti [69] proposed the construction of self-similar Markov processes in a following way where we will follow the notations of Bertoin and Yor [17]. Let $\xi = (\xi_t)_{t\geq 0}$ be a real-valued Lévy process which does not drift to $-\infty$, i.e. $\limsup_{t\to\infty} \xi_t = +\infty$ a.s. First, implicitly define $\tau = (\tau_t)_{t\geq 0}$ by the identity

$$t = \int_0^{\tau(t)} \exp(\xi_s) ds, \quad t \geq 0,$$

and then for an arbitrary $x > 0$, define the process $X^{(x)}$ started from x at time $t = 0$ by

$$X_t^{(x)} := x \exp\{\xi_{\tau(t/x)}\}, \quad t \geq 0.$$

The family of processes $(X^{(x)}, x > 0)$ is Markovian and self-similar, since there is the scaling identity $X_t^{(x)} \stackrel{d}{=} xX_{t/x}^{(1)}$. Conversely, any Markov process on $(0, \infty)$ with the scaling property can be constructed like this.

We recall from Section 4.3.1 that ξ can be characterized by its characteristic exponent Ψ, which admits the following Lévy-Khintchine representation

$$\Psi(u) = \frac{\sigma^2}{2}u^2 + bu + \int_{-\infty}^{\infty} (e^{ux} - 1 - ux1_{\{|x|<1\}})\Pi(dx), \tag{4.21}$$

The condition of finite exponential moments is equivalent to $\int_1^\infty e^{ux}\Pi(dx) < \infty$ for all $u > 0$, which holds for instance when the jumps of ξ is bounded above by some fixed number, and, in particular, include the spectrally negative case (see Sato [106, Theorem 25.17]). We will then have

$$\mathbb{E}[e^{u\xi_t}] = e^{t\Psi(u)} < \infty, \quad t, u \geq 0. \tag{4.22}$$

In this direction, the condition that ξ does not drift to $-\infty$ is equivalent to

$$m = \mathbb{E}[\xi_1] = \Psi'(0+) \in [0, \infty). \tag{4.23}$$

Moreover, note that by the well-studied Wiener-Hopf factorization, see e.g. [67, Section 6], Ψ can be decomposed by

$$\Psi(u) = -\phi_+(-u)\phi_-(u), \quad u \geq 0,$$

where $\phi_\pm$ both are Bernstein functions necessarily taking the form

$$\phi_\pm(z) = \kappa_\pm + \gamma_\pm z + \int_0^\infty (1 - e^{-zy})\Pi_\pm(dy), \tag{4.24}$$

with $\kappa_\pm \geq 0$ such that $\kappa_+\kappa_- = 0$, $\gamma_\pm \geq 0$ and $\Pi_\pm$ being a Lévy measure which satisfies the integrability condition $\int_0^\infty (y \wedge 1)\Pi_\pm(dy) < \infty$. One shall note that, on one hand, $0 < \lim_{u\to\infty} |\phi_+(-u)| < \infty$ only when $\phi_+ = \kappa_+$ is a constant; on the other hand, $0 < \lim_{u\to\infty} |\phi_-(u)| < \infty$ only when ϕ_- is the Laplace exponent of a compound Poisson process. Therefore, $\lim_{u\to\infty} \Psi(u) < \infty$ only when ξ is a decreasing compound Poisson process, and we can exclude this case in our discussion. Therefore, we will assume $\lim_{u\to\infty} \Psi(u) = +\infty$ in the rest of this paper.

Proposition 4.3.2. *Assume that ξ is not a compound Poisson process. Let $K = (K_t)_{t\geq 0}$ be the semigroup of the 1-self-similar Markov process $X = (X_t)_{t\geq 0}$ associated to Ψ defined by* (4.21)*. Assume that* (4.22) *and* (4.23) *hold, and define*

$$\mathfrak{I}_\Psi(z) = \sum_{n=0}^{\infty} a_n(\Psi) z^n, \tag{4.25}$$

where $a_0(\Psi) = 1$ and $a_n(\Psi) = \frac{1}{\prod_{k=1}^n \Psi(k)}$ for $n \geq 1$. Then, we have the following.

(i) $\mathfrak{I}_\Psi$ is an entire function. Moreover, it is positive and increasing on $\mathbb{R}_+$.

(ii) For any $q > 0$, $d_q \mathfrak{I}_\Psi \in \mathcal{I}_q(K)$.

(iii) For any $q > 0$,

$$K_t^* f(x) := K_t(f \circ d_q \mathfrak{I}_\Psi)(x) \tag{4.26}$$

defines a q-pricing semigroup.

Proof. To show (i), first observe that for all $n \in \mathbb{N}$, we have $\frac{|a_{n+1}(\Psi)|}{|a_n(\Psi)|} = \frac{1}{\Psi(n+1)}$. Hence, recalling that we excluded the case of the negative of compound Poisson processes, the analyticity of $\mathfrak{I}_\Psi$ follows from the fact that $\lim_{u\to\infty} \Psi(u) = +\infty$. Moreover, using a dominated convergence argument similar to [106, Lemma 26.4], we see that $\Psi''(u) = \sigma^2 + \int_{-\infty}^{\infty} x^2 e^{ux} \Pi(dx) \geq 0$, and $\Psi''(u) = 0$ only if $\sigma = 0$ and $\Pi \equiv 0$, which means X is a pure drift and this case is excluded from our discussion. Hence $\Psi''(u) > 0$ and Ψ is a purely convex function. Meanwhile, under the assumption (4.23) that $\Psi'(0+) \geq 0$, we must have $\Psi'(u) > 0$ for all $u \geq 0$. Therefore, Ψ is a non-decreasing function. Moreover, since $\Psi(0) = 0$, we see that $\Psi(u) > 0$ for all $u > 0$, which means $a_n(\Psi) > 0$ for all $n \geq 0$. Therefore, $\mathfrak{I}_\Psi$ is an entire function with all positive coefficients, and, in particular, positive and monotone increasing on $\mathbb{R}_+$. To show (ii), note that if (4.22) and (4.23) hold, then by [17, Proposition 1], we have for every $t \geq 0$ and $n \geq 0$,

$$K_t p_n(x) = \sum_{k=0}^{n} \frac{\Psi(n)\cdots\Psi(n-k+1)}{k!} t^k p_{n-k}(x) = \frac{1}{a_n(\Psi)} \sum_{k=0}^{n} \frac{a_{n-k}(\Psi)}{k!} t^k p_{n-k}(x). \tag{4.27}$$

Now, for any $q > 0$, using the definition of $\Im_\Psi$ in (4.25), (4.27) and applying Fubini's theorem, we have

$$\begin{aligned} K_t d_q \Im_\Psi(x) &= \sum_{n=0}^{\infty} a_n(\Psi) q^n K_t p_n(x) = \sum_{n=0}^{\infty} q^n \sum_{k=0}^{n} a_{n-k}(\Psi) \frac{t^k}{k!} p_{n-k}(x) \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} q^{n-k} a_{n-k}(\Psi) \frac{(qt)^k}{k!} p_{n-k}(x) = \sum_{k=0}^{\infty} \frac{(qt)^k}{k!} \sum_{n=0}^{\infty} a_n(\Psi) d_q p_n(x) = e^{qt} d_q \Im_\Psi(x), \end{aligned}$$

where we used that for any sequence $(a_{n,k})_{n,k\geq 0}$, the following holds

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} a_{n,k} = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} a_{n+k,k}. \tag{4.28}$$

Therefore, $e^{-qt} K_t d_q \Im_\Psi(x) = d_q \Im_q \Psi(x)$, hence $d_q \Im_\Psi \in \mathcal{I}_q(K)$.

Part (iii) directly follows from Corollary 4.2.1. □

Remark 4.3.2. *Proposition 4.3.2 can be generalized for $\frac{1}{\alpha}$-self-similar processes for any $\alpha > 0$. That is, if $K = (K_t)_{t\geq 0}$ is the semigroup associated to the $\frac{1}{\alpha}$-self-similar Markov process $X = (X_t)_{t\geq 0}$, and* (4.22) *and* (4.23) *hold, then for any $q \geq 0$, $\Im^q_{\Psi,\alpha}(x) := \Im_{\Psi,\alpha}(qx^\alpha)$ is a q-invariant function for the semigroup K where*

$$\Im_{\Psi,\alpha}(x) = \sum_{n=0}^{\infty} a_n(\Psi, \alpha) x^n, \quad x \geq 0$$

with $a_n(\Psi, \alpha) = \frac{1}{\prod_{k=1}^{n} \Psi(\alpha k)}$.

Special sub-classes of self-similar processes have many applications in financial modeling, and next we talk we talk about Bessel processes.

Bessel processes

Definition 4.3.2.1. *Let $W = (W^{(1)}, W^{(2)}, \cdots, W^{(m)})$ be an m-dimensional Brownian motion. The process $R^{(\nu)}$ defined by*

$$R^{(\nu)}_t := (W^{(1)}_t)^2 + \cdots + (W^{(m)}_t)^2, \quad t \geq 0,$$

is called a squared Bessel process of dimension m, or equivalently, of order/index $\nu = m/2 - 1$.

Clearly, if we take the radial part of an n-dimensional Brownian motion, the resulting diffusion belongs to the family of squared Bessel processes. In other words, Bessel process is constructed as the squared distance between the origin and the position of the m-dimensional Brownian motion at time t.

Using Itô's formula, we get

$$R_t^{(\nu)} = mdt + 2\sqrt{R_t^{(\nu)}}dW_t^*,$$

where W_t^* is a one-dimensional Brownian motion constructed from W_t as

$$W_t^* = \sum_{i=1}^{n} \int_0^t \frac{W_s^{(i)}}{\sqrt{R_s^{(\nu)}}} dW_s^{(i)}.$$

Next, we can provide an interpretation of Lampertis relation in terms of the mappings we introduced earlier in this chapter. The latter states that, for any fixed ν, there exists a Brownian motion B such that one has

$$e^{B_t^\nu} = R^{(\nu)}\left(\int_0^t e^{B_s^\nu} ds\right)$$

where $e^{B_t^\nu} = B_t + \nu t$ for any $t \geq 0$.

The infinitesimal generator of the squared Bessel process $R_t^{(\nu)}$ of dimension m is given by

$$\mathbf{A}f(x) = 2xf''(x) + 2(1+\nu)f'(x).$$

It is well known that a squared Bessel process is a continuous-path 1-self-similar strong Markov processes on $[0, \infty)$ (see Lamperti [69]). Hence, we can apply the risk-neutralization techniques for self-similar processes discussed above and in particular, for squared Bessel processes too. Note that in this case, we have

$$\Psi(u) = \frac{1}{2}u^2 + \nu u,$$

and $a_n(\Psi) = \prod_{k=1}^{n} \frac{1}{2}k^2 + \nu k = \frac{1}{2^n} n! \frac{\Gamma(n+1+2\nu)}{\Gamma(1+2\nu)}$. Hence the function

$$\mathfrak{I}_q(x) = \Gamma(1+2\nu) \sum_{n=0}^{\infty} \frac{(2qx)^n}{n!\Gamma(n+1+2\nu)} = \frac{\Gamma(1+2\nu)}{(2qx)^\nu} I_{2\nu}(2\sqrt{2qx})$$

where $I_{2\nu}$ denotes the modified Bessel function of the first kind of order 2ν, is a q-invariant function for the squared Bessel semigroup of order ν.

Bessel processes emerge in many financial problems and have remarkable properties. It plays an essential role for evaluating Asian option and contingent claims under the CIR model. In many financial applications, the calculation of the first time a diffusion process reaches a certain level is important, as for instance in the case of barrier options.

4.3.3 Generalized Cox-Ingersoll-Ross Models

The generalized Cox-Ingersoll-Ross (CIR) or generalized Laguerre (gL) processes are intimately connected to the so-called generalized Ornstein-Uhlenbeck processes, and hence our interest to look at these processes from the financial applications viewpoint.

We say a semigroup $P = (P_t)_{t\geq 0}$ is a generalized CIR semigroup if it can be represented as

$$P_t f(x) = K_{e^t - 1} f \circ d_{e^{-t}}(x), \quad x > 0, \tag{4.29}$$

for some $K = (K_t)_{t\geq 0}$ being a 1-self-similar semigroup defined in Section 4.3.2, where we recall that $d_c f(x) = f(cx)$ is the dilation operator. We say that a process

$X = (X_t)_{t\geq 0}$ defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ is a generalized CIR process if the family of linear operators $P = (P_t)_{t\geq 0}$ defined, for any $t \geq 0$ and $f \in C_0(\mathbb{R}_+)$, by $P_t f(x) = \mathbb{E}_x[f(X_t)]$, is a generalized CIR semigroup. To see that this class of processes is indeed a generalization of CIR processes, one can equivalently define $P = (P_t)_{t\geq 0}$ to be the semigroup of generalized CIR process, if writing $P_t = e^{t\mathbf{A}}$, we have fore f smooth on $x > 0$,

$$\mathbf{A}f(x) = \sigma^2 x f''(x) + (m + \sigma^2 - x) f'(x) + x \int_0^\infty f''(xy) M(y) dy, \tag{4.30}$$

where $\sigma, m \geq 0$, and $M = \int_{-\ln y}^{\infty} \overline{\Pi}(r) dr$ with $\overline{\Pi}$ being the tail of the Lévy measure. From Section 4.3.2, we already know that any self-similar Markov process can be constructed from a real-valued Lévy process $\xi = (\xi_t)_{t\geq 0}$ with characteristic exponent $\Psi : i\mathbb{R} \to \mathbb{C}$ defined by (4.21). Hence by the deterministic relation (4.29) between P and K, we see that there is a bijection between each generalized CIR semigroup P (or its corresponding process) and the function Ψ. Assume that $\Psi(n) < \infty$, for some $n \in \mathbb{N}$, then with $p_n(x) = x^n$,

$$\mathbf{A}p_n(x) = \Psi(n) p_{n-1}(x) - n p_n(x).$$

Note that if $M = \overline{\Pi} = 0$, then P os the semigroup of the CIR process. Now, again assume ξ has finite exponential moments of any positive order, and it does not drift to $-\infty$, i.e. (4.22) and (4.23) hold. For any $\alpha > 0$, define the function F_q as

$$F_q(z) = \sum_{n=0}^{\infty} a_n(\Psi; \alpha)(q)_n z^n, \quad z \in \mathbb{C}, \tag{4.31}$$

with $a_0 = 1, a_n(\Psi; \alpha)^{-1} = \prod_{k=1}^{n} \Psi(\alpha k)$, and $(q)_n = \frac{\Gamma(q+n)}{\Gamma(q)}$ is the Pochhammer symbol. Then we have the following results.

Proposition 4.3.3. *If* (4.22) *and* (4.23) *hold, and further assuming that* $\lim_{u\to\infty} \frac{\Psi(u)}{u} = R$*, then for any* $q \geq 0$*, the following statements hold.*

(i) The function F_q defined by (4.31) *is analytic on $|z| < R$, and is non-negative and non-decreasing on* $(0, R)$*. Moreover, $F_q \in \mathcal{I}_q(P)$.*

(ii) The family of operators $Q = (Q_t)_{t\geq 0}$ defined by

$$Q_t f(y) = P_t(f \circ F_q)(x), \quad y = F_q(x),$$

is a q-pricing semigroup.

Proof. First, for $q \geq 0$, note that

$$\frac{|a_{n+1}(\Psi;\alpha)(q)_{n+1}|}{|a_n(\Psi;\alpha)(q)_n|} = \frac{q+n}{\Psi(\alpha(n+1))}.$$

Since we have that $\lim_{u\to\infty} \frac{\Psi(u)}{u} = R$, it follows that F_q is analytic function on $|z| < R$. Moreover, observe that $F_q'(x) = \sum_{n=0}^{\infty} n a_n(\Psi;\alpha)(q)_n x^{n-1}$. Due to the non-negativeness of Ψ, $a_n(\Psi;\alpha) \geq 0$ for all n and so does $(q)_n$, hence F_q is non-decreasing on $(0, R)$. On the other hand, since $F_q(0) = 0$, we see that $F_q(x) \geq 0$ for all $x \in (0, R)$. Hence F_q is also non-negative on $(0, R)$. For the second part, we will demonstrate the proof for the case $\alpha = 1$ with notation $a_n(\Psi; 1) = \frac{1}{\prod_{k=1}^{n} \Psi(k)} = a_n(\Psi)$ without loss of generality, since the case for general $\alpha > 0$ follows similarly. First, we apply Tonelli's theorem to change the order of integration and get

$$P_t F_q(x) = \sum_{n=0}^{\infty} a_n(\Psi)(q)_n P_t p_n(x) = \sum_{n=0}^{\infty} a_n(\Psi)(q)_n K_{e^t-1} p_n \circ d_{e^{-t}}(x), \tag{4.32}$$

where we have successively used the linearity of P_t and the identity (4.29). With a change of variable $e^t - 1 = s$ and hence $e^{-t} = \frac{1}{1+s}$, we now have

$$P_t F_q(x) = \sum_{n=0}^{\infty} a_n(\Psi)(q)_n K_s p_n \circ d_{\frac{1}{1+s}}(x) = \sum_{n=0}^{\infty} a_n(\Psi)(q)_n (1+s)^{-n} K_s p_n(x). \tag{4.33}$$

Now, since (4.23) and (4.22) hold, we can substitute (4.27) in (4.33) to get

$$\begin{aligned}
P_tF_q(x) &= \sum_{n=0}^{\infty} a_n(\Psi)(q)_n(1+s)^{-n}K_s p_n(x) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{\Gamma(q+n)}{\Gamma(q)}(1+s)^{-n}a_{n-k}(\Psi)p_{n-k}(x)\frac{s^k}{k!} \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n} a_{n-k}(\Psi)\frac{\Gamma(q+n-k)}{\Gamma(q)}p_{n-k}(x)\frac{1}{(1+s)^{n-k}}\frac{1}{(1+s)^k}\binom{q+n-1}{k}s^k \\
&= \sum_{n=0}^{\infty} a_n(\Psi)\frac{\Gamma(q+n)}{\Gamma(q)}p_n(x)\frac{1}{(1+s)^n}\sum_{k=0}^{\infty}\binom{q+n+k-1}{k}\left(\frac{s}{1+s}\right)^k \\
&= \sum_{n=0}^{\infty} a_n(\Psi)\frac{\Gamma(q+n)}{\Gamma(q)}x^n\frac{1}{(1+s)^n}\frac{1}{\left(1-\frac{s}{1+s}\right)^{q+n}} = (1+s)^qF_q(x) = e^{qt}F_q(x),
\end{aligned}$$

where for the second last identity, we used the following binomial formula for which the radius of convergence of the series is 1,

$$\frac{1}{(1-z)^{c+1}} = \sum_{n=0}^{\infty}\binom{n+c}{n}z^n$$

where $\binom{c}{k} = \frac{c(c-1)\cdots(c-k+1)}{k(k-1)\cdots 1}$ for $k \in \mathbb{N}$ and arbitrary $c \in \mathbb{C}$ are the generalized Binomial coefficients. Therefore $F_q \in \mathcal{I}_q(P)$, and, consequently, the semigroup $Q = (Q_t)_{t\geq 0}$ defined by $Q_tf(x) = P_t(f \circ F_q)(x)$ is a q-pricing semigroup. Finally, item (ii) is a direct application of Corollary 4.2.1, and this concludes the proof. □

Besides the risk-neutralization techniques presented in this paper, we recall from [91] that one can also deduce a spectral expansion for P_tf under certain circumstances. In particular, when $\Pi(dx) \equiv 0$ on $(0,\infty)$, which means X only has negative jumps, then P admits an invariant measure, that is, a positive measure with an absolutely continuous density $\nu(x)$, such that $\nu P_tf = \nu f$ for all $t \geq 0$ and $f \in L^2(\nu)$. Moreover, for f in a proper subspace of $L^2(\nu)$, we have the spectral expansion of P_tf

$$P_tf = \sum_{n=0}^{\infty} e^{-nt}\langle f, \mathcal{V}_n\rangle_\nu \mathcal{P}_n$$

where, for all $n \geq 0$,

$$\mathcal{P}_n(x) = \sum_{k=0}^{n}(-1)^k\frac{\binom{n}{k}}{W_\phi(k+1)}x^k \in L^2(\nu),$$

with $W_\phi(1) = 1$ and $W_\phi(n+1) = \prod_{k=1}^n \phi(k), n \geq 1$, is an eigenfunction of P_t with eigenvalue e^{-nt}, and

$$\mathcal{V}_n(x) = \frac{\mathcal{R}^{(n)}\nu(x)}{\nu(x)} = \frac{(x^n\nu(x))^{(n)}}{n!\nu(x)} \in L^2(\nu),$$

in a co-eigenfunction of P_t (eigenfunction for the dual semigroup) with eigenvalue e^{-nt}. In the following section, we will present an example to illustrate the usefulness of our risk-neutralization techniques, and moreover, show that these techniques can be combined with the spectral expansion to yield powerful results.

Small perturbation of Laguerre process

One specific instance of generalized CIR semigroups is the small perturbation of the Laguerre semigroup. Below we'll give the description of this process, and later we'll do some pricing of European options and show a numerical illustration. Let $\mathfrak{m} \geq 1$ and consider, for any $u > 0$,

$$\phi_\mathfrak{m}(u) = \frac{(u+\mathfrak{m}+1)(u+\mathfrak{m}-1)}{u+\mathfrak{m}} = u + \frac{\mathfrak{m}^2-1}{\mathfrak{m}} + \int_0^\infty (1-e^{-uy})e^{-\mathfrak{m}y}dy.$$

From this we can also get that

$$\psi(u) = u\frac{(u+\mathfrak{m}+1)(u+\mathfrak{m}-1)}{u+\mathfrak{m}}.$$

The infinitesimal generator of the associated generalized CIR semigroup is the integro-differential operator

$$\mathbf{A}_\mathfrak{m} f(x) = xf''(x) + \left(\frac{\mathfrak{m}^2-1}{\mathfrak{m}} + 1 - x\right)f'(x) + \frac{\mathfrak{m}}{x}\int_0^\infty (f(e^{-y}x) - f(x) + yxf'(x))e^{-\mathfrak{m}y}dy, \tag{4.34}$$

for at least functions in $\mathcal{D} = \{f; x \mapsto f_e(x) = f(e^x) \in \mathrm{C}^2([-\infty,\infty])\}$. Moreover, $W_{\phi_\mathfrak{m}}(n+1) = \frac{\mathfrak{m}}{n+\mathfrak{m}}\frac{\Gamma(n+\mathfrak{m}+2)}{\Gamma(\mathfrak{m}+2)}$, that is, by moment identification,

$$\nu(x) = \frac{1+x}{\mathfrak{m}+1}\frac{x^{\mathfrak{m}-1}e^{-x}}{\Gamma(\mathfrak{m})} = \frac{(1+x)}{\mathfrak{m}+1}\varepsilon_{\mathfrak{m}-1}(x), \quad x > 0,$$

where for any α with $-\alpha \notin \mathbb{N}$, $\varepsilon_\alpha(x) = \frac{x^\alpha e^{-x}}{\Gamma(\alpha+1)}$, $x > 0$. For $n \geq 1$, the $\mathcal{P}_n$'s and $\mathcal{V}_n$'s can be expressed in terms of the Laguerre polynomials as follows,

$$\mathcal{P}_n(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{\Gamma(\mathfrak{m}+2)}{\Gamma(\mathfrak{m}+k+2)} \frac{\mathfrak{m}+k}{\mathfrak{m}} x^k, \tag{4.35}$$

$$\mathcal{V}_n(x) = \frac{1}{x+1} \mathcal{L}_n^{(\mathfrak{m}-1)}(x) + \frac{x}{x+1} \mathcal{L}_n^{(\mathfrak{m})}(x), \tag{4.36}$$

where $\mathfrak{c}_n(\mathfrak{m}+1) = \frac{\Gamma(n+1)\Gamma(\mathfrak{m}+2)}{\Gamma(n+\mathfrak{m}+2)}$ and for any $-\alpha \notin \mathbb{N}$, $\mathcal{L}_n^{(\alpha)}$ is the n-th associated Laguerre polynomial (or generalized Laguerre polynomial) of order α, defined either by means of the Rodrigues operator $\mathcal{R}^{(n)}$ as follows

$$\mathcal{L}_n^{(\alpha)}(x) = \frac{\mathcal{R}^{(n)} \varepsilon_\alpha(x)}{\varepsilon_\alpha(x)} = \frac{1}{n!} \frac{(x^n \varepsilon_\alpha(x))^{(n)}}{\varepsilon_\alpha(x)} \quad x > 0,$$

or, through the polynomial representation

$$\mathcal{L}_n^{(\alpha)}(x) = \sum_{k=0}^{n} (-1)^k \binom{n+\alpha}{n-k} \frac{x^k}{k!}.$$

We also have that for all $f \in L^2(\nu)$ and $t > 0$,

$$P_t f(x) = \sum_{n=0}^{\infty} e^{-nt} \langle f, \mathcal{V}_n \rangle_\nu \mathcal{P}_n(x).$$

Moreover, by Proposition 4.3.3, we can easily write the q-invariant function as

$$\begin{aligned} F_q(x) &= \sum_{n=0}^{\infty} a_n(\psi_\mathfrak{m})(q)_n x^n = \sum_{n=0}^{\infty} \frac{1}{\prod_{k=1}^{n} \psi_\mathfrak{m}(k)} (q)_n x^n \\ &= \sum_{n=0}^{\infty} \prod_{k=1}^{n} \frac{k+\mathfrak{m}}{k(k+\mathfrak{m}+1)(k+\mathfrak{m}-1)} (q)_n x^n = \sum_{n=0}^{\infty} \frac{(\mathfrak{m}+1)_n (q)_n}{(\mathfrak{m}+2)_n (\mathfrak{m})_n} \frac{x^n}{n!} \\ &= {}_2F_2(\mathfrak{m}+1, q, \mathfrak{m}+2, \mathfrak{m}; x), \end{aligned}$$

where ${}_2F_2$ is the generalized hypergeometric function. Figure 4.1 shows a plot of F_q under different $\mathfrak{m}$'s (assuming a constant risk-free rate at $r = 0.03$).

Using this process, we can also provide an example for Corollary 4.2.2. First, observe that for each $n \geq 0$, $(q)_n$ is a polynomial of q with order n, hence for any

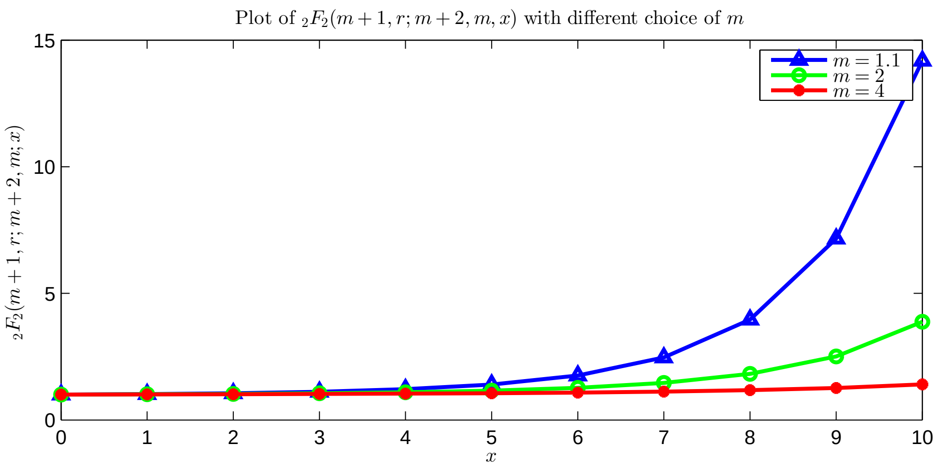

Figure 4.1: Plot for $r = 0.03, \mathfrak{m} = 1.1, 2, 4$

fixed x, $q \mapsto F_q(x)$ is an entire function of q. Moreover, since $F_q(0) = 1$ and by [86, Theorem 2.1], we have $\mathbb{E}_0[e^{-qT_x}] = \frac{1}{F_q(x)}$, where $T_x = \inf\{t \geq 0 : X_t = x\}$ is the first hitting time of x, we have that, for any fixed $x \geq 0$, the mapping $q \mapsto F_q(x)$ is, as the reciprocal of a Laplace transform, holomorphic and zero-free on the right-half plane $\mathbb{C}_+$ and its first zero, if any, should be located on $\mathbb{R}_-$. Let us take $x = a$ for some $a > \frac{\mathfrak{m}(\mathfrak{m}+2)}{\mathfrak{m}+1}$, and let ζ_1 denote the first (negative) zero of $q \mapsto F_q(a)$, then we make the following observations. On the one hand, we have $F_q(a) > 0$ for all $q \geq 0$ and on the other hand, $F_{-1}(a) = 1 - \frac{\mathfrak{m}+1}{\mathfrak{m}(\mathfrak{m}+2)}a < 0$. Therefore, we must have $\zeta_1 \in (-1, 0)$, which implies that $\Gamma(\zeta_1) < 0$ while $\Gamma(\zeta_1 + n) > 0$ for all $n \geq 1$. Furthermore, since

$$F'_{\zeta_1}(x) = \frac{1}{\Gamma(\zeta_1)} \sum_{n=1}^{\infty} \frac{(\mathfrak{m}+1)_n \Gamma(\zeta_1 + n)}{(\mathfrak{m}+2)_n (\mathfrak{m})_n} \frac{x^{n-1}}{(n-1)!} < 0,$$

we see that $F_{\zeta_1}(x)$ is decreasing on $(0, a)$, and this combined with the fact that $F_{\zeta_1}(a) = 0$ yields the conclusion that $F_{\zeta_1}(x)$ is a positive and monotonically decreasing function on $(0, a)$.

Let us consider the semigroup $P_t^{\dagger} f(x) = \mathbb{E}_x[f(X_t), t < T_a]$, i.e. the semigroup of the process X killed at a. Its infitesimal generator is $\mathbf{A}_{\mathfrak{m}}$ as given in (4.34), for at least functions in $f \in \mathcal{D}$ satisfying the boundary condition $f(a) = 0$. In order to show that $F_{\zeta_1}(x)$ is an eigenfunction of $P_t^\dagger$, we simply apply $\mathbf{A}_{\mathfrak{m}}$ as shown in (4.34) to F_{ζ_1}. Note that all derivatives and integrations can be applied term-by-term, and we get

$$\mathbf{A}_{\mathfrak{m}} F_{\zeta_1}(x) = \zeta_1 F_{\zeta_1}(x),$$

and $F_{\zeta_1}(a) = 0$ is trivial by definition of ζ_1. Equivalently, we have $P_t^\dagger F_{\zeta_1}(x) = e^{\zeta_1 t} F_{\zeta_1}(x)$. We now proceed by proving that $F_q^\dagger \in \mathcal{I}_q(P^\dagger)$, where $F_q^\dagger(x) = F_q(x)\mathbf{1}_{\{x<a\}}$. To this end, we observe that for any $x < a$,

$$\begin{aligned}\mathcal{G}_{\mathfrak{m}} F_q^\dagger(x) &= \lim_{t\to 0} \frac{P_t^\dagger F_q(x) - F_q(x)\mathbb{P}_x(T_a > t)}{t} = \lim_{t\to 0} \frac{P_t F_q(x) - F_q(x)\mathbb{P}_x(T_a > t) - \mathbb{E}_x[F_q(X_t)\mathbf{1}_{\{t>T_a\}}]}{t} \\ &= \lim_{t\to 0} \frac{e^{qt} F_q(x) - F_q(x)\mathbb{P}_x(T_a > t)}{t} - \lim_{t\to 0} \frac{\mathbb{E}_x[F_q(X_t)\mathbf{1}_{\{t>T_a\}}]}{t} = qF_q(x) = qF_q^\dagger(x),\end{aligned}$$

where we successively used the facts that $F_q \in \mathcal{I}_q(P)$ and $\lim_{t\to 0} \frac{\mathbb{E}_x[F_q(X_t)\mathbf{1}_{\{t>T_a\}}]}{t} = 0$. Combined with the fact that $F_q^\dagger$ is strictly increasing on $(0, a)$, we see that, the function $\mathfrak{s} = \frac{F_q^\dagger}{F_{\zeta_1}}$ is monotonically increasing on $(0, a)$ and thus it is a homeomorphism. Hence, the semigroup Q as defined by (4.16) is a $(q - \zeta_1)$-pricing semigroup, where we recall that $\zeta_1 < 0$.

4.4 Pricing Derivatives

In this section we want to implement our transformation techniques to some financial models. As we know, the price of a contingent claim can be expressed as the expectation, under the risk-neutral measure, of the payoff discounted to present value. More precisely, we consider a contract of European type, which specifies a payoff $V(S_T)$, depending on the level of the underlying asset S_t at maturity $t = T$. The value V of such contract at time $t = 0$, conditional to an underlying value S_0 is

$$V(S_0) = \mathbb{E}[e^{-rT} V(S_T)],$$

where $\mathbb{E}$ denotes the expectation under the risk-neutral measure and r is the risk-free rate. Particularly, let us suppose the market is driven by a small perturbation of the Laguerre process $X = (X_t)_{t \geq 0}$, which we recall was studied in Section 4.3.3, and we model the price process as $S_t = F_r(X_t)$ and the current spot is $S_0 = F_r(x)$. Consider a European put option with strike K and maturity T, then the option value is given by

$$\begin{aligned} P(S_0, K, r, T; \mathfrak{m}) &= \mathbb{E}_{F_r(x)}\left[e^{-rT}(K - S_T)^+\right] = e^{-rT}\mathbb{E}_x\left[(K - F_r(X_T))^+\right] \\ &= e^{-rT}\sum_{n=0}^{\infty} e^{-nT}\left\langle (K - F_r)^+, \mathcal{V}_n \right\rangle_\nu \mathcal{P}_n(x). \end{aligned}$$

Note that the inner product $\langle (K - F_r)^+, \mathcal{V}_n \rangle_\nu$ is easy to evaluate numerically. On the other hand, in order to evaluate the rate of convergence, we introduce, for $N = 1, 2, \ldots$ the N-th order spectral approximate for the option price

$$\mathcal{S}_N = \sum_{n=0}^{N} e^{-nT}\left\langle (K - F_r)^+, \mathcal{V}_n \right\rangle_\nu \mathcal{P}_n(x). \tag{4.37}$$

We also introduce the following quantity, for $\epsilon > 0$,

$$\mathcal{N}_\epsilon = \inf\{N \geq 4;\ \max\{|\mathcal{S}_N - \mathcal{S}_{N-1}|, |\mathcal{S}_N - \mathcal{S}_{N-2}|, |\mathcal{S}_N - \mathcal{S}_{N-3}|\} \leq \epsilon\}. \tag{4.38}$$

That is, $\mathcal{N}_\epsilon$ is the smallest number of terms needed in the spectral expansion such that the truncated series has "converged" in the sense that the $(\mathcal{N}_\epsilon - 1)$-th, $(\mathcal{N}_\epsilon - 2)$-th and $(\mathcal{N}_\epsilon - 3)$-th order truncated summation do not differ from the $\mathcal{N}_\epsilon$-th by more than ϵ. To illustrate a numerical example, we evaluate an at-the-money put option with $S_0 = K = 10, r = 0.03, T = 1$ and $\mathfrak{m} = 4$, with tolerance level $\epsilon = 10^{-4}$. Our algorithm returns the option price $P(S_0, K, r, T; \mathfrak{m}) \approx \mathcal{S}_{\mathcal{N}_{10^{-4}}} =$

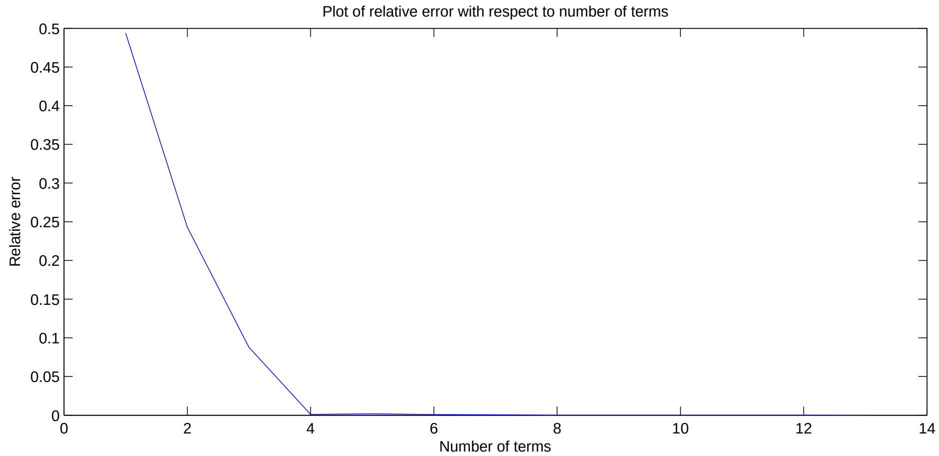

Figure 4.2: Relative series truncation error for $S_0 = K = 10, r = 0.03, T = 1, \mathfrak{m} = 4$

5.9571. The relative truncation error at order k, which is denoted by $e(k)$, and defined as

$$e(k) = \frac{\left|\sum_{n=0}^{k} e^{-nT} \langle (K - F_r)^+, \mathcal{V}_n \rangle_\nu \mathcal{P}_n(x) - \mathcal{S}_{\mathcal{N}_{10^{-4}}}\right|}{\mathcal{S}_{\mathcal{N}_{10^{-4}}}},$$

is shown in Figure 4.2. Moreover, in Table 4.1 the value of $\mathcal{N}_\epsilon$ for $\epsilon = 10^{-4}$ are computed for various strikes and expiry times. The approximated option values are shown in Table 4.2.

Maturity T \ Strike K	7	10	13	17
0.33	23	23	23	22
0.5	17	18	18	18
1	11	11	11	11
2	7	7	7	7

Table 4.1: Values of $\mathcal{N}_{10^{-4}}$ for various K and T with parameters $S_0 = 10, r = 0.03, \mathfrak{m} = 4$.

Maturity T \ Strike K	7	10	13	17
0.33	3.656	6.176	8.831	12.443
0.5	4.314	6.984	9.745	13.457
1	5.152	7.969	10.831	14.634
2	5.438	8.241	11.070	14.814

Table 4.2: Values of $\mathcal{S}_{\mathcal{N}_{10^{-4}}}$ for various K and T with parameters $S_0 = 10$, $r = 0.03$, $\mathfrak{m} = 4$.

BIBLIOGRAPHY

[1] N. I. Akhiezer. *The classical moment problem and some related questions in analysis*. Translated by N. Kemmer. Hafner Publishing Co., New York, 1965.

[2] J.-P. Antoine and P. Balazs. Frames and semi-frames. *J. Phys. A*, 44(47):479501, 2, 2011.

[3] W. Arendt. Does diffusion determine the body? *J. Reine Angew. Math.*, 550:97–123, 2002.

[4] W. Arendt, M. Biegert, and A. F. M. ter Elst. Diffusion determines the manifold. *J. Reine Angew. Math.*, 667:1–25, 2012.

[5] W. Arendt, A. F. M. ter Elst, and J. B. Kennedy. Analytical aspects of isospectral drums. *Oper. Matrices*, 8(1):255–277, 2014.

[6] B. Baeumer, M. Kovács, M. M. Meerschaert, R. L. Schilling, and P. Straka. Reflected spectrally negative stable processes and their governing equations. *Trans. Amer. Math. Soc.*, 368(1):227–248, 2016.

[7] Dominique Bakry, Ivan Gentil, and Michel Ledoux. *Analysis and geometry of Markov diffusion operators*, volume 348 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Cham, 2014.

[8] M. Barczy and L. Döring. On entire moments of self-similar Markov processes. *Stoch. Anal. Appl.*, 31(2):191–198, 2013.

[9] P. Bérard. Transplantation et isospectralité. I. *Math. Ann.*, 292(3):547–559, 1992.

[10] P. Bérard. Transplantation et isospectralité. II. *J. London Math. Soc. (2)*, 48(3):565–576, 1993.

[11] C. Berg and G. Forst, Potential theory on locally compact abelian groups. *Springer-Verlag*, 1975

[12] V. Bernyk, R. C. Dalang, and G. Peskir. Predicting the ultimate supremum of a stable Lévy process with no negative jumps. *Ann. Probab.*, 39(6):2385–2423, 2011.

[13] J. Bertoin. An extension of Pitman's theorem for spectrally positive lévy processes. *Ann. Probab.*, 20(3):1464–1483, 1992.

[14] J. Bertoin. *Lévy processes*, volume 121 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1996.

[15] J. Bertoin, T. Budd, N. Curien, and I. Kortchemski. Martingales in self-similar growth-fragmentations and their connections with random planar maps. *Submitted*, 2016.

[16] J. Bertoin, N. Curien, and I. Kortchemski. Random planar maps & growth-fragmentations. *Ann. Prob, to appear*, 2017.

[17] J. Bertoin and M. Yor. On the entire moments of self-similar Markov processes and exponential functionals of Lévy processes. *Ann. Fac. Sci. Toulouse Math. (6)*, 11(1):33–45, 2002.

[18] J. Bertoin and M. Yor. Exponential functionals of Lévy processes. *Probab. Surv.*, 2:191–212, 2005.

[19] J. Bertoin and M. Yor. *The entrance laws of self-similar Markov processes and exponential functionals of Lévy processes*. *Potential Analysis*, 17: 389400, 2002

[20] R. M. Blumenthal. *On construction of Markov processes*, volume 63 of *Z. Wahrsch. Verw. Gebiete*. 63(4):433–444, 1983.

[21] R. C. Blumenthal, M. Robert and R. K. Getoor. Markov processes and potential theory. Courier Corporation, 2007.

[22] S. Bochner. Harmonic analysis and the theory of probability. *Courier Corporation*, 2012.

[23] A. N. Borodin and P. Salminen. *Handbook of Brownian Motion - Facts and Formulae*. Springer, 2012.

[24] J. Borovička, L. P. Hansen and J. Scheinkman. Misspecified recovery. *The Journal of Finance*, 71(6):2493–2544, 2016.

[25] B. Böttcher, R. Schilling and J. Wang. Lévy matters. III *Lecture Notes in Mathematics*, volume 2099, xviii+199, 2013.

[26] L. M. Bujorianu Stochastic reachability analysis of hybrid systems. Springer Science & Business Media, 2012.

[27] C. Burnap and P. F. Zweifel. A note on the spectral theorem. *Integral Equations Operator Theory*, 9(3):305–324, 1986.

[28] P. Carr, H. Geman, D. B. Madan and M. Yor. The fine structure of asset returns: An empirical investigation. *The Journal of Business*, volume 75(2), 305–332, 2002.

[29] P. Carr and J. Yu. Risk, return, and Ross recovery. *The Journal of Derivatives*, volume 20(1), 38–59, 2012.

[30] Ph. Carmona, F. Petit, and M. Yor. On the distribution and asymptotic results for exponential functionals of Lévy processes. *In M. Yor (ed.) Exponential functionals and principal values related to Brownian motion. Biblioteca de la Rev. Mat. Iberoamericana*, pages 73–121, 1997.

[31] Ph. Carmona, F. Petit, and M. Yor. Sur les fonctionnelles exponentielles de certains processus de lévy. *Stochastics An International Journal of Probability and Stochastic Processes*, 47:71–101, 1994.

[32] Ph. Carmona, F. Petit, and M. Yor. Beta-gamma random variables and intertwining relations between certain Markov processes. *Rev. Mat. Iberoamericana.*, 14, 1998.

[33] M. Chazal, A. E. Kyprianou, and P. Patie. A transformation for Lévy processes with one-sided jumps and applications. *Adv. Appl. Prob., to appear*.

[34] O. Christensen. *An Introduction to Frames and Riesz Bases*. Birkhäuser, 2003.

[35] K. L. Chung and J. B. Walsh. *Markov processes, Brownian motion, and time symmetry*, volume 249 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, New York, second edition, 2005.

[36] E. Çinlar, J. Jacod, P. Protter and M.J. Sharpe. Semimartingales and Markov Processes. *Z. Wahrscheinlichkeitstheorie verw. Gebiete*, volume 54, 161–219, 1980.

[37] T. Craven and G. Csordas. Jensen polynomials and the Turán and Laguerre inequalities. *Pacific J. Math.*, 136(2):241–260, 1989.

[38] G. Da Prato. *An introduction to infinite-dimensional analysis*. Universitext. Springer-Verlag, Berlin, 2006. Revised and extended from the 2001 original by Da Prato.

[39] F. Delbaen and W. Schachermayer A general version of the fundamental theorem of asset pricing, *Mathematische annalen.*, 300(1):463–520, 1994.

[40] C. Dellacherie and P. A. Meyer. Probabilities and Potential, C: Potential Theory for Discrete and Continuous Semigroups. Elsevier, 2011.

[41] E. B. Davies. *Linear operators and their spectra*, volume 106 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2007.

[42] P. Diaconis and J.A. Fill. Strong stationary times via a new form of duality. *Ann. Probab.*, 18(4):1483–1522, 1990.

[43] C. Donati-Martin, B. Roynette, P. Vallois, and M. Yor. On constants related to the choice of the local time at 0, and the corresponding itô measure for bessel processes with dimension $d = 2(1-\alpha), 0 < \alpha < 1$. *Studia Scientiarum Mathematicarum Hungarica*, 45-2:207–221, 2008.

[44] S. Dubynskiy and R. S. Goldstein. Recovering drifts and preference parameters from financial derivatives., 2013

[45] N. Dunford and J. T. Schwartz. *Linear operators. Part II: Spectral theory. Self adjoint operators in Hilbert space*. With the assistance of William G. Bade and Robert G. Bartle. Interscience Publishers John Wiley & Sons New York-London, 1963.

[46] N. Dunford and J. T. Schwartz. *Linear operators. Part III: Spectral operators*. Interscience Publishers [John Wiley & Sons, Inc.], New York-London-Sydney, 1971. With the assistance of William G. Bade and Robert G. Bartle, Pure and Applied Mathematics, Vol. VII.

[47] E. B. Dynkin. Markov processes. Vol. I. *Translated with the authorization and assistance of the author by J. Fabius, V. Greenberg, A. Maitra, G. Majone. Die Grundlehren der Mathematischen Wissenschaften, Bände 121*, volume 122, 1965.

[48] V. J. Ervin and J.P. Roop. Variational formulation for the stationary frac-

tional advection dispersion equation. *Numer. Methods Partial Differential Equations*, (22):558–576, 2006.

[49] J.A. Fill. On hitting times and fastest strong stationary times for skip-free and more general chains. *J. Theoret. Probab.*, 22(3):587–600, 2009.

[50] P. J. Fitzsimmons. On the existence of recurrent extensions of self-similar markov processes. *Electron. Comm. Probab.*, 11:230–241, 2006.

[51] P. J. Fitzsimmons and R. K. Getoor. Excursion theory revisited. *Illinois J. Math.*, 50(1-4):413–437, 2006.

[52] M. B. Garman. Towards a semigroup pricing theory. *The Journal of Finance*, 40(3):847–861, 1985.

[53] R. K. Getoor. *Excessive measures*. Probability and its Applications. Birkhäuser Boston, Inc., Boston, MA, 1990.

[54] R. K. Getoor. Measure perturbations of Markovian semigroups. *Potential Anal.*, 11(2):101–133, 1999.

[55] C. Gordon, D. L. Webb, and S. Wolpert. One cannot hear the shape of a drum. *Bull. Amer. Math. Soc. (N.S.)*, 27(1):134–138, 1992.

[56] H. Gzyl. Infinitesimal generators of time changed processes. *Ann. Probab.*, 8(4):716–726, 1980.

[57] J. M. Harrison and S. R. Pliska. Martingales and stochastic integrals in the theory of continuous trading. *Stochastic processes and their applications.* 11(3): 215–260, 1981.

[58] F. Hirsch and M. Yor. On the remarkable Lamperti representation of the inverse local time of a radial Ornstein-Uhlenbeck process. *Bull. Belg. Math. Soc. Simon Stevin*, 20(3):435–449, 2013.

[59] K. Itô. Poisson point processes attached to Markov processes. *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability (Univ. California, Berkeley, Calif., 1970/1971), Vol. III: Probability theory*, 225–239, 1972.

[60] M. Kac. Can one hear the shape of a drum? *Amer. Math. Monthly*, 73(4, part II):1–23, 1966.

[61] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo. *Theory and applications of fractional differential equations*, volume 204 of *North-Holland Mathematics Studies*. Elsevier Science B.V., Amsterdam, 2006.

[62] S. Kotani. On a generalized Sturm-Liouville operator with a singular boundary. *J. Math. Kyoto Univ.*, 15(2):423–454, 1975.

[63] S. Kotani and S. Watanabe. Kreĭn's spectral theory of strings and generalized diffusion processes. In *Functional analysis in Markov processes (Katata/Kyoto, 1981)*, volume 923 of *Lecture Notes in Math.*, pages 235–259. Springer, Berlin-New York, 1982.

[64] D. M. Kreps. Arbitrage and equilibrium in economies with infinitely many commodities. *Journal of Mathematical Economics*, 8(1):15–35, 1981.

[65] U. Küchler and P. Salminen. On spectral measures of strings and excursions of quasi diffusions. 1372:490–502, 1989.

[66] A. Kuznetsov and M. Kwasnicki. Spectral analysis of stable processes on the positive half-line. *available at http://arxiv.org/abs/1509.06435*, 2015.

[67] A. E. Kyprianou. *Fluctuations of Lévy processes with applications*. Universitext. Springer, Heidelberg, second edition, 2014. Introductory lectures.

[68] A. E. Kyprianou and P. Patie. A Ciesielski-Taylor type identity for positive self-similar Markov processes. *Ann. Inst. Henri Poincaré Probab. Stat.*, 47(3):917–928, 2011.

[69] J. Lamperti. Semi-stable Markov processes. I. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 22:205–225, 1972.

[70] N. N. Lebedev. *Special functions and their applications*. Dover Publications Inc., New York, 1972. Revised edition, translated from the Russian and edited by Richard A. Silverman, Unabridged and corrected republication.

[71] L. Li and V. Linetsky. Time-Changed Ornstein–Uhlenbeck Processes and Their Applications in Commodity Derivative Models. *Mathematical Finance*, volume 24(2), 289–330, 2014.

[72] B. Maisonneuve. Exit systems. *Ann. Probability*, 3(3):399–411, 1975.

[73] H. P. McKean, Jr. Elementary solutions for certain parabolic partial differential equations. *Trans. Amer. Math. Soc.*, 82:519–548, 1956.

[74] M. M. Meerschaert and A. Sikorskii. *Stochastic models for fractional calculus*, volume 43 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, 2012.

[75] R. Mendoza-Arriaga,P. Carr and V. Linetsky. Time-changed Markov processes in unified credit-equity modeling. *Mathematical Finance*, volume 20(4), 527–569, 2010.

[76] R. C. Merton. On the Pricing of Corporate Debt: The Risk Structure of Interest Rates. *The Journal of Finance*, volume 29(2), 449-470, 1974.

[77] O. P. Misra and J. L. Lavoine. *Transform analysis of generalized functions*, volume 119 of *North-Holland Mathematics Studies*. North-Holland Publishing Co., Amsterdam, 1986. Notas de Matemática [Mathematical Notes], 106.

[78] B. Muckenhoupt and E. M. Stein. Classical expansions and their relation to conjugate harmonic functions. *Trans. Amer. Math. Soc.*, 118:17–92, 1965.

[79] A. Nowak and K. Stempak. On L^p-contractivity of Laguerre semigroups. *Illinois J. Math.*, 56(2):433–452, 2012.

[80] F. W. J. Olver. *Introduction to Asymptotics and Special Functions*. Academic Press, 1974.

[81] Analysis of continuous strict local martingales via h-transforms. *Stochastic Processes and their Applications*, volume 120(8), 1424–1443, 2010.

[82] R. B. Paris. Exponentially small expansions in the asymptotics of the Wright function. *J. Comput. Appl. Math.*, 234(2):488–504, 2010.

[83] R. B. Paris and D. Kaminski. *Asymptotics and Mellin-Barnes Integrals*. Cambridge University Press, 2001.

[84] H. Park. The Martin Integral Representation of Markovian Pricing Kernels. *arXiv preprint arXiv:1504.00276*, 2015.

[85] P. Patie. A refined factorization of the exponential law. *Bernoulli*, 17(2):814–826, 2011.

[86] P. Patie. Infinite divisibility of solutions to some self-similar integro-differential equations and exponential functionals of Lévy processes. *Ann. Inst. Henri Poincaré Probab. Stat.*, 45(3):667–684, 2009.

[87] P. Patie and M. Savov. Spectral theory of positive self-similar semigroups. *Working paper*, 2015.

[88] P. Patie and M. Savov. Cauchy problem of the non-self-adjoint Gauss-Laguerre semigroups and uniform bounds of generalized Laguerre polynomials. *J. Spectral Theory, to appear,* 33p., 2016.

[89] P. Patie and M. Savov. Exponential functional of Lévy processes: generalized Weierstrass products and Wiener-Hopf factorization. *C. R. Math. Acad. Sci. Paris*, 351(9-10):393–396, 2013.

[90] P. Patie and M. Savov. Bernstein-gamma functions and exponential functionals of Lévy Processes. *Submitted*, 85p., 2016.

[91] P. Patie and M. Savov. Spectral expansion of non-self-adjoint generalized Laguerre semigroups. *Submitted*, page 162p. (current version), 2016.

[92] P. Patie and T. Simon. Intertwining certain fractional derivatives. *Potent. Anal.*, 36:569–587, 2012.

[93] P. Patie and Y. Zhao. Spectral decomposition of fractional operators and a reflected stable semigroup. *J. Differential Equations*, 262(3):1690–1719, 2017.

[94] J. Pitman and M. Yor. Decomposition at the maximum for excursions and bridges of one-dimensional diffusions. In *Itô's stochastic calculus and probability theory*, pages 293–310. Springer, Tokyo, 1996.

[95] A. D. Poularikas, editor. *The transforms and applications handbook*. The Electrical Engineering Handbook Series. CRC Press, Boca Raton, FL; IEEE Press, New York, 1996.

[96] P. Protter. *Stochastic integration and differential equations*, volume 21 of *Applications of Mathematics (New York)*. Springer-Verlag, Berlin, 1990. A new approach.

[97] L. Qin and V. Linetsky. Positive eigenfunctions of Markovian pricing operators: Hansen-Scheinkman factorization, Ross recovery and long-term pricing. *arXiv preprint arXiv:1411.3075.*, 2014.

[98] L. Qin, and V. Linetsky. Long-Term Risk: A Martingale Approach. *Econometrica*, volume 85(1), 299–312, 2017.

[99] D. Revuz. Mesures associées aux fonctionnelles additives de Markov. I. *Trans. Amer. Math. Soc.*, 148:501–531, 1970.

[100] D. Revuz and M. Yor. *Continuous martingales and Brownian motion*, volume 293 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, third edition, 1999.

[101] V. Rivero. Recurrent extensions of self-similar Markov processes and Cramér's condition. *Bernoulli*, 11(3):471–509, 2005.

[102] L. C. G. Rogers. Itô excursion theory via resolvents. *Z. Wahrsch. Verw. Gebiete*, 63(2):237–255, 1983.

[103] L. C. G. Rogers and J. W. Pitman. Markov functions. *The Annals of Probability*, 573–582, 1981.

[104] S. Ross. The recovery theorem. *The Journal of Finance*, volume 70(2), 615–648, 2015.

[105] H. Sankaranarayanan. *Grünwald-type approximations and boundary conditions for one-sided fractional derivative operators*. PhD thesis, University of Otago, 2014.

[106] K.-I. Sato. *Lévy processes and infinitely divisible distributions*, 1999.

[107] R. L. Schilling, R. Song, and Z. Vondraček. *Bernstein functions*, volume 37 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, 2010. Theory and applications.

[108] H. Schmidli. Stochastic control in insurance. Springer Science & Business Media, 2007.

[109] N. Sekeljic. Asymptotic expansion of bessel functions: Applications to electromagnetics. *Dynamics at the Horsetooth*, 2A, 2010.

[110] J. Sjöstrand. Some results on nonselfadjoint operators: a survey. In *Further progress in analysis*, pages 45–74. World Sci. Publ., Hackensack, NJ, 2009.

[111] T. Sunada. Riemannian coverings and isospectral manifolds. *Ann. of Math. (2)*, 121(1):169–186, 1985.

[112] P. Tankov. *Financial modelling with jump processes*, 2003.

[113] E. C. Titchmarsh. *The theory of functions*. Oxford University Press, Oxford, 1939.

[114] E. C. Titchmarsh. *Introduction to the theory of Fourier integrals*. Chelsea Publishing Co., New York, third edition, 1986.

[115] J. Walden. Recovery with unbounded diffusion processes. *Review of Finance*, 2017.

[116] J. B. Walsh. Markov processes and their functionals in duality. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 24:229–246, 1972.